Making up Numbers
A History of Invention in Mathematics

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CHAPTER 5

Struggles with the Infinite

*Again there is another great and powerful cause why the sciences have made but little progress; which is this. It is not possible to run a course aright when the goal itself has not been rightly placed.*

Sir Francis Bacon, *Novum Organum*, 1620

**Summary**

In this chapter we review how mathematicians (and sometimes philosophers) of previous centuries dealt with the troublesome concept of *infinity*. Our overview must necessarily be concise – this is not a full historical account by any means! We will focus, instead, on two key periods, nearly 2000 years apart.

For the first we return to Ancient Greece to consider Aristotle’s conception of the *potential infinite* and the difficulties that notions of infinite divisibility of space and time presented. Next come the works of Archimedes, also transmitted via the Arab world, with their remarkably sophisticated comparisons of the areas and volumes of various curvilinear figures. One might echo Descartes’ suspicions (mentioned in Chapter 3) ‘that these writers then with a sort of low cunning, deplorable indeed, suppressed’ their methods for discovering these relationships. In fact, Archimedes’ recently rediscovered letter, *The Method of Mechanical Theorems*, addressed to his friend Erastosthenes, shows how he had used his *law of the lever*, together with ‘infinitesimal slices’ of solid bodies and areas, to arrive at his results. These ingenious techniques, foreshadowing arguments used nearly two millennia later in the Calculus, did not conform to the rigorous standards of proof of Euclidean geometry. In his public tracts Archimedes stated his results and proved them by *contradiction*, often generalising the ‘method of exhaustion’ established earlier by Eudoxus, with no explanation how he had discovered the relationships his proofs verified.

Finally we consider the development of the Calculus from the late seventeenth century onward, including the serious logical issues it posed. We focus on the different conceptions of the two principal contributors, Isaac Newton and Gottfried Wilhelm Leibniz. While they are rightly celebrated
as inventors of the Calculus, both relied on versions of a controversial ‘principle of continuity’, expressed by Leibniz as ‘whatever succeeds for the finite, also succeeds for the infinite’. The logical difficulties this created are seen most directly in the methods of proof employed in the eighteenth century in the prolific writings of Leonhard Euler – his results were almost always correct, but many could only be verified rigorously in the nineteenth century. His treatment of the ‘natural logarithm’ function is a classic example.

1. Zeno and Aristotle

Differing perceptions of the nature and role of infinity in mathematics have pervaded the subject since the era of Ancient Greece. Aristotle insisted that mathematicians have no need of actual infinity, but can work with the unlimited, the potentially infinite. For example, the natural numbers may be regarded as a potentially infinite collection, since the process of counting them can in theory be continued indefinitely, even though, in practice, we cannot continue counting forever (as I found out and reported in the Prologue). His perceptions, although highly influential, have not always held sway.

Early in the fifth century BCE, the philosopher Parmenides of Elea reasoned that nothing can ever change, because ‘nothing comes from nothing’. In support of this view, Zeno (490-430 BCE) presented a number of famous paradoxes. He wished to show that motion is logically impossible, and is therefore a sensory illusion. To do this, he presented arguments that challenge the notion of the ‘infinite divisibility’ of space and time. His paradoxes stimulated much discussion among the Pre-Socratic Greek philosophers, and were addressed at some length by Aristotle and other later commentators.

The most famous of his examples is popularly known as Achilles and the Tortoise, although Zeno merely asserts that in a race the quickest runner cannot overtake the slowest if the latter has a head start. In the popular version, Achilles, the mythical fastest runner in antiquity, cannot overtake a slow tortoise: to reach the tortoise he must first pass the tortoise’s starting point A, by which time the tortoise has moved to some point B, further on. When Achilles reaches B the tortoise is at some point C beyond B, and so on indefinitely. Thus, while the tortoise’s lead becomes ever smaller, after any finite number of these stages it is still ahead. To overtake the tortoise, Achilles would have to cover an infinite number of intervals in a finite time, which is impossible, Zeno argued.

Aristotle struggled to refute Zeno’s reasoning conclusively. He argued that Achilles moves in a ‘continuous’ motion, thereby implicitly conceding

\[^1\]
Aristotle renders the claim as follows in his Physics: In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.
that time and space can be regarded as potentially infinitely divisible. He says that an infinite number of intervals can be covered in a finite time, since ‘while a thing in a finite time cannot come in contact with things quantitatively infinite, it can come in contact with things infinite in respect to divisibility, for in this sense time itself is also infinite’. This does not explain how we might compute the instant at which Achilles will overtake the tortoise. Today, this can be done with an apparently (!) simple calculation.

To fix ideas, assume that the tortoise has a head start of 10 units and moves at 1 unit per minute, while Achilles is 10 times as quick, so that he covers each unit in $\frac{1}{10}$ th minute. Achilles reaches the tortoise’s starting point in 1 minute, by which time the tortoise has moved on 1 unit. Achilles covers that unit in the next $\frac{1}{10}$ th minute, at the end of which the tortoise is now only $\frac{1}{10}$ th unit ahead, which Achilles then covers in $\frac{1}{10} \times \frac{1}{10} = \frac{1}{10^2}$ minutes, and so on.

Hence the time (in minutes) it takes Achilles to catch up with the tortoise is given by the sum of the infinite geometric series

\[
1 + \frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^n} + \ldots
\]

However, this is a series with infinitely many terms, so we need to understand what we mean by its sum. For any finite $n$ we can sum the first $n$ terms, and consider what happens when we let $n$ ‘grow’. But we need to decide what this last phrase should mean.

More generally, let $-1 < x < 1$ and set $S_n = 1 + x + x^2 + \ldots + x^{n-1}$. We call $S_n$ the $n^{th}$ partial sum of the series $1 + x + x^2 + \ldots + x^n + \ldots$. If we now multiply both sides by $(1 - x)$, we find that

\[
(1 - x)S_n = (1 - x)(1 + x + x^2 + \ldots + x^{n-1}) = 1 - x^n,
\]

as the inner terms in the product cancel in pairs. This is the archetypal example of a ‘telescoping sum’. So we have

\[
S_n = \frac{1 - x^n}{1 - x} = \frac{1}{1 - x} - \frac{x^n}{1 - x}.
\]

For fixed $x < 1$, $\frac{1}{1-x} > 0$ is constant, and the final term will behave like $x^n$. We claim that $x^n$ can be made as close to 0 as we please by taking $n$ large enough – a formal proof of this will be given in Chapter 7. Assuming this for now, we see that the sequence of these partial sums $(S_n)_{n \geq 1}$ therefore gets ever closer to $\frac{1}{1-x}$. This ‘limiting value’ is now taken as the sum of the infinite series.

When $x = \frac{1}{10}$, we obtain $\frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$, so that Achilles catches up with the tortoise after $\frac{10}{9}$ minutes.

This kind of quantitative analysis was not available to Aristotle. He rejected the Pythagorean claim that ‘All is Number’ and argued that there
are two types of quantities, distinguishing between discrete *multiples* (of the unit) which could be represented by whole numbers (where each number has an immediate successor), and continuous *magnitudes* (where there are no immediate successors). Multitudes could be handled with arithmetic, while magnitudes belonged to the domain of geometry. For Aristotle, objects like line segments with a common endpoint touch each other and, as he puts it, ‘the touching limits of each become one and the same’. He argues that a continuous line, a *continuum*, should not be seen as an aggregate of individual ‘points’. Successive division of a line segment into two equal parts produces not points, but ever shorter line segments, which eventually become smaller than any pre-assigned magnitude.

Aristotle insists that continuous motion ‘flows’ along the line and cannot be described as going from point to point in succession. Thus points only have potential, not actual, existence and adding them together does not produce a line segment. In his *Physics* he argues that division of a continuous line into two halves makes the original midpoint into an endpoint in each half, which destroys continuity, both of the motion and of the line. The line can be halved repeatedly to produce an unbounded number of such successively shorter halves, not in reality but only potentially. It is a process whose end result is never fully actualised. For Aristotle, *continuity* of motion (as yet undefined mathematically) is the key concept.

The emphasis on describing these ideas by means a static analysis of shapes, as in plane and solid geometry, may explain why the Ancient Greek mathematicians had difficulty in describing motion mathematically. Aristotle insisted that a given velocity achieved by a moving body must *persist for a time*, and rejected notions of ‘instantaneous’ change. This, together with the distinction between the discrete and continuous, meant that Greek mathematicians were not able to deal effectively with variable motion.

The key stimulus that would provide a solution to these problems was the gradual development of the Calculus in the sixteenth and seventeenth centuries – although, as will be seen next, a good deal of this was foreshadowed in a work of Archimedes which remained lost for nearly a millennium and was not analysed fully until quite recently. Between 1200 and 1600, thinkers in various parts of Europe extensively debated Aristotle’s views when discussing variable motion as well as the nature of space and time. The notion of the infinite divisibility of space and time gradually gained ground. Unknown to them, many of their ideas had been foreshadowed by Archimedes.
A century after Aristotle, Archimedes developed tools for the calculation of areas and volumes of various curvilinear bodies (sphere, cone, cylinder, parabolic segment, spirals, etc.), leading to startling new numerical relationships between these objects.

Two typical examples, one comparing volumes and one comparing areas, illustrate the remarkable sophistication and scope of his results:

The first is from his treatise *On Conoids and Spheroids*:

**Theorem**

The volume of a segment of a paraboloid of revolution cut off by a plane at right angles to the axis (we might think of this as a ‘bullet’) is in the ratio 3:2 to that of the cone (think ice-cream cone!) which has the same base and axis. (See Figure 25.)

In his surviving works, probably housed first in the library of Alexandria, later translated or transcribed in Baghdad and Constantinople before reaching the West some 1500 years after his death, Archimedes stated many results like the above, and invariably verified the formulae he discovered by extending the method of exhaustion, due to Eudoxus more than a century earlier and used extensively by Euclid. This method involves using inscribed and circumscribed figures, most often regular polygons or circular segments, whose properties were well understood, fitting inside and outside the given shape. The inscribed and circumscribed figures are then successively modified (typically by bisecting the sides of regular polygons) in order to approximate the desired shape ever more closely. One then confirms the truth of the given formula with a proof by contradiction—assume that one side of the equation to be verified is greater than the other, then show that, in finitely many steps, we will arrive at a claim that contradicts a known property of the approximating figures.
In works such as *On Conoids and Spheroids* Archimedes gives no indication how he arrived at his results in the first place—after all, a proof by contradiction requires him to assume that the ratio to be verified (3 : 2 in this case) is incorrect and then to show that this leads to a contradiction.

However, in a letter entitled *The Method of Mechanical Theorems* (now usually simply called *The Method*) and addressed to Eratosthenes, Archimedes described clearly how he used his well-known law of the lever to compare infinitesimal slices of solid bodies to arrive at the formulae he then proceeded to verify painstakingly, using (and extending) Eudoxus’ method of exhaustion. The argument in the shaded paragraph below shows how he relates the volume of the circular cone to that of the cut-off paraboloid with the same base, as stated above.

In Figure 25, consider the segment \( BAC \) of a parabola with vertex at \( A \) and cut off by the line \( BC \). We compare segment \( BAC \) with the rectangle \( CBEF \) and with the triangle \( ABC \). Draw \( AD \) parallel to \( FC \) and \( EB \), so that, by the symmetry of the parabola, \( D \) is the midpoint of \( BC \). Rotating all three plane shapes—the triangle \( ABC \), the parabolic segment and the rectangle \( CBEF \)—through a full revolution about the line \( AD \) produces a circular cone, a paraboloid of revolution (which we called a ‘bullet’ in the statement of the theorem) and a cylinder, respectively. The cone will have volume equal to \( \frac{1}{3} \) that of the cylinder, as was well-known in Archimedes’ time.

Now extend \( DA \) to \( H \) so that \( DA = AH \), and, from now on, treat \( A \) as the fulcrum of a lever. Archimedes imagines himself ‘weighing’ infinitesimal slices of the ‘bullet’—represented by lines in our two-dimensional figures—against slices of the cylinder, placed where they are on \( DA \). Choosing any point \( P \) on the parabolic segment, he draws the line \( GPSM \) parallel to \( CB \) to meet \( CF \) in \( G \), the parabolic segment in \( P \), \( DA \) in \( S \) and \( BE \) in \( M \).

By its definition the parabola satisfies the proportion \( DA : AS = (CD)^2 : (PS)^2 \). The line segments \( PS, SM \), rotated about \( S \) on \( DA \), produce circles with radii \( PS \) and \( GS = CD \) respectively. As circles are to one another as the square on their radii, he concludes that \( DA : AS \) is the same ratio as the area of the circle in the cylinder is to that of the circle in the paraboloid of revolution. But \( HA = DA \) so the ratio \( HA : AS \) has the same property.

So the circle of radius \( GS \) in the cylinder (thought of as an infinitesimal slice), placed where it is, will balance the circle of radius \( PS \) if the latter is placed at \( H \). But we may regard the weight of the whole cylinder as being placed at its centre of gravity, which is the midpoint \( K \) of \( AD \). Archimedes imagines placing the centres of all the circular slices of the cylinder at \( K \), and ‘balances’ them against the totality of slices of the parabolic segment, all placed at \( H \).

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2 The law of the lever states that on a scale with a central fulcrum and two linear arms (like a seesaw) the scale will balance precisely when the product of the weight placed on one side and its distance from the fulcrum equals the same product on the other side.
In *Conoids and Spheroids* he had shown that equality of ratios for all the individual pieces implies equality of the sums of \( n \) such ratios taken on each side, for every finite \( n \). Now he asserts (without proof) that this also holds for sums of infinitely many ratios, and applies this to the sums of his ‘slices’.

Consequently, by the law of the lever, the ratio \( HA : AK \) represents the ratio of the cylinder and the segment of the paraboloid of revolution! Since \( AK = \frac{1}{2} HA \), the volume of the parabolic segment is half that of the cylinder, and, since the cylinder has volume three times that of the cone, the volume of the segment is \( \frac{3}{2} \) times that of the cone, as he had claimed.

*Today we would write this as \( y^2 = 4ax \), meaning in particular that the change in the \( y \)-direction is proportional to the square of the change in the \( x \)-direction.*

Archimedes’ use of his ‘law of the lever’ shows that he treated the volume of the curved bodies he compares as proportional to their weight. He also assumes that the total volume of his infinitely many ‘infinitely thin’ slices will equal the volume of the whole body. These techniques, while entirely plausible, do not conform to the rigorous demands of Euclidean geometry. This explains why Archimedes then goes on to prove by the ‘method of exhaustion’ that the ratios he has identified using this technique are the correct ones.

On the other hand, as a means for discovering what these ratios must be, his informal arguments are clearly very productive. His use of summing infinitely many infinitesimal slices was well ahead of its time. It was eventually reinvented independently some 1800 years later to produce the integral calculus for the computation of areas and volumes.

As a second example of the fruits of Archimedes’ novel techniques we briefly mention a result from his later work *On Spirals:*

*The area of the first full turn of the spiral is \( \frac{1}{3} \) of the area of the circle whose radius is the distance between the origin \( O \) of the spiral and the point \( P \) reached at the first full turn.*

(In Figure 26, the shaded area is \( \frac{1}{3} \) of the area of the circle with centre \( O \) and radius \( OP \)).

The Archimedean spiral is defined as the locus of a point, starting from \( O \), which moves uniformly along a line \( OA \), which is itself rotating uniformly about \( O \). Archimedes puts it as follows:

‘*If a straight line, one extremity of which remains fixed, be made to revolve at a uniform rate in the plane until it returns to its starting position and, if at the same time as the straight line is revolving, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane.*’
We express this using polar coordinates (see Chapter 4): \( r = a\phi \) for some \( a > 0 \), where \( \phi \) is the angle at \( O \) that \( OA \) makes with its original position. At \( P \) in Figure 26 we have \( \phi = 360^\circ \). The constant \( a \) is the ratio of the constant velocity of the linear motion of the point and the constant angular velocity of rotation of the line. Archimedes’ quite involved constructions will be omitted here (see [19]).

Archimedes’ use of the concept of the locus of a point involves motion, but the velocity of the moving points and lines used explain how to trace out complex figures (which typically cannot be drawn by rules and compass alone), is always assumed to be constant. Mathematical descriptions of the motion of accelerated objects, one of the key features of applied Calculus, had to wait for Isaac Newton.

One possible reason why it took so long may be that for fully a thousand years, between the tenth and twentieth centuries, Archimedes’ Method disappeared from view, so that its contents were not available to Renaissance and seventeenth-century mathematicians, who had to reconstruct them from scratch, with much painful effort and over an extended period.

The Method seems to have had no influence on Islamic geometry, although many Greek manuscripts were studied minutely by scholars in Baghdad and Constantinople from the eighth century to the eleventh century. It could have been available to them, however. In the sixth century, Isidorus of Miletus, the architect of the Hagia Sophia church in Constantinople, collected, into a single document, letters by Archimedes previously held in the Great Library of Alexandria. This included what we now know as The Method. Around the middle of the tenth century an unknown scribe copied it onto a parchment that then found its way to Jerusalem by the thirteenth century, where the text was partially erased and overwritten by monks with Christian liturgical text and lost to science. There is no evidence that any copies of
The Method were transmitted to Europe during the Renaissance. The heavily overwritten palimpsest was discovered in a monastery and brought back to Constantinople around 1840, where it was catalogued.\(^3\) Around the turn of the century it was studied in situ by the eminent Danish historian Johan Heiberg, who confirmed Archimedes’ authorship. Amid the tumult of the Greco-Turkish war that followed World War I, the palimpsest then disappeared once more. It finally resurfaced at an auction in 1998, was bought for $2.2 million, and is now held in Walter’s Art Museum, Baltimore, USA.\(^4\)

3. Infinitesimals in the calculus

Mediaeval thinkers, meanwhile, had spent a good deal of effort on the problem of infinite divisibility of lines. For example, Thomas Bradwardine (1295-1349), Archbishop of Canterbury, argued that the line is made up of infinitely many indivisible segments, but distinguished these ‘atoms’ from ‘points’. He argued that the atoms were magnitudes of the same kind as the line they produced – in line with Aristotle’s thinking. This viewpoint allows ‘infinitesimals’ only a potential existence: the continued division of the line in arbitrarily many steps only produces ever shorter lines, not points.

Such arguments did little to resolve some of the obvious paradoxes of the infinite that gained more attention in Europe during the Middle Ages. For example, two concentric circles share the same lines for their radii, so the ‘atoms’ of their respective circumferences can be ‘paired off’ exactly, each pair consisting of the two points where a given radius meets the two circumferences. Yet the circumferences of the two circles are clearly not the same!

A paradox along similar lines is contained in the famous Dialogue concerning two New Sciences (1638) by Galileo Galilei (1564-1642): among (positive) whole numbers, some are perfect squares (the ‘square numbers’ of Chapter 1) and some are not. Therefore there should be more positive whole numbers than square numbers. Yet, for each \( n \geq 1 \), the positive whole number \( n \) can be paired uniquely with the \( n^\text{th} \) perfect square \( n^2 \). So the collections of positive whole numbers and the perfect squares can be ‘paired off’ exactly. Despite evidence that this example had been known for some time

\(3\) A palimpsest is a document where earlier writing has been partially erased and overwritten.

\(4\) The later history of the Archimedes palimpsest is bizarre. During the 1920s it was acquired by a French traveller in the Middle East, having fairly recently been overpainted with gold leaf by a forger. It then spent more than 60 years (many of them in a mouldy cellar) with his family, who, initially unaware of its significance, had sought a private buyer for several years before putting it up for auction. After 1998, the text was studied extensively, translated and finally published by Cambridge University Press in 2011 as The Archimedes Palimpsest, (2 vols.). Reviel Netz, one of the authors, claims (perhaps with a degree of hyperbole) that the palimpsest reveals that the work of the Western scientific revolution since the seventeenth century is, in essence, simply ‘a series of footnotes to Archimedes’.
before Galileo, his fame ensured that this observation became widely known as *Galileo’s Paradox*. It was variously discussed until the late nineteenth century.

Examples like these led to an early realisation that there are difficulties in handling *infinite aggregates* arithmetically. Nevertheless, notions about the usefulness of the infinitely small and infinitely large in mathematics increasingly found adherents, especially in connection with theological speculations and a renewed interest in Plato’s philosophy, which theologians such as *St. Augustine of Hippo* (354-430) had earlier sought to reconcile with biblical dogma.

### 3.1. The Principle of Continuity

*Nicholas of Cusa* (1401-1461), born as Nikolaus Krebs in what is now Berncastel-Kues on the Moselle, became arguably the most influential German theologian and philosopher of the fifteenth century, serving as papal legate to Germany for much of his later career. He illustrated his notion of the ‘coincidence of opposites’, which he sees in the relationship between between God and Man, with various mathematical metaphors, arguing that:

(i) by continuously increasing the number of sides of a polygon, we will eventually reach a circle,

(ii) by increasing the radius of a circle indefinitely, the tangent at a point becomes identical with the circumference,

(iii) although the centre and circumference of a circle are opposites, by shrinking the radius until it is infinitesimal, these opposites coincide.

In this way he sought to reconcile the apparent contradiction of the finiteness of our world and the infinite being of God: the diversity and multiplicity of our finite existence become one in the realm of God, who both transcends and resides in every part of the universe.

His cosmology, based on these precepts, was remarkably prescient, although not based on any direct evidence. He believed that the universe has no ‘centre’ (rejecting prevalent geocentric doctrines well before Copernicus): for him, the universe and its centre are the same.\(^5\) Nor is the Earth at rest: ‘It is impossible for the world machine to have this sensible earth, air, fire, or anything else for a fixed and immovable centre. For in motion there is simply no minimum, such as a fixed centre.... And although the world is not Infinite, it cannot be conceived of as finite, since it lacks boundaries within which it is enclosed. ... Therefore,

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\(^5\)In *De Docta ignorantia* he writes: *Life, as it exists on Earth in the form of men, animals and plants, is to be found, let us suppose in a high form, in the solar and stellar regions. Rather than think that so many stars and parts of the heavens are uninhabited and that this earth of ours alone is peopled – and that with beings perhaps of an inferior type – we will suppose that in every region there are inhabitants, differing in nature by rank and all owing their origin to God, who is the centre and circumference of all stellar regions*
just as the earth is not the centre of the world, so the sphere of fixed stars is not its circumference.’

Nicholas’ thinking influenced the work of major mathematical figures such as Kepler, Leibniz and Euler, over the next three centuries. Johannes Kepler (1571-1630), for example, imagined the sphere of radius r as made up of an infinite number of infinitely thin cones with their vertices at the centre of the sphere, and bases on the surface of the sphere. He took their bases to be small enough to allow him to assume that the flat base of the cone and the surface area of the corresponding part of the sphere are the same. He then calculated the volume of the sphere to be $\frac{4}{3}\pi r^3$. For this, he applied two facts well-known since Ancient Greece: the volume of each cone is $\frac{1}{3}$ of the product of its base and its height $r$, while the sum of the (infinitely many!) bases is the surface area of the sphere, which the Greeks had shown to be $4\pi r^2$.

Similarly, the Italian mathematician Bonaventura Cavalieri calculated the areas of various curved bodies by imagining their areas as made up of an infinite number of line segments (regarded alternatively, as ‘infinitely thin’ slices, to ensure that each had the same dimension as the total figure, as Aristotle had demanded) and summing this infinite collection to find the desired area.

3.2. Leibniz. In the 1670s, Gottfried Wilhelm Leibniz (1646-1716) formulated his version of the ideas initiated by Nicholas of Cusa as a formal Principle of Continuity. One may regard it essentially as an operational maxim: ‘whatever succeeds for the finite, also succeeds for the infinite’ – although, in terms of his mathematical ideas, he may well be referring to the infinitely small rather than the infinitely large! Expressing his principle more formally, he writes in 1701:

‘In any supposed continuous transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included’.

Leibniz used this principle to justify his extensive use of infinitesimal quantities, which he employed to compute curvilinear areas (see Figure 28). This enabled him to turn Cavalieri’s ‘method of indivisibles’ into a technique for finding the areas under various types of (‘smooth’ enough) curves.

Crucially, he realised early on in his studies that finding the area under a curve (integration) and determining the tangent to the curve (differentiation) were opposites, or inverse operations. A series of manuscripts in which he noted down his developing insights in late 1675 suggests how this came about.6

6This very brief summary draws on the essay Newton, Leibniz and the Leibnizian Tradition by Henk Bos, in [17].
The origin of his discovery lay in the simple relationship between a sequence of numbers \((a_i)_{i \geq 1}\) and the sequence \((b_j)_{j \geq 1}\) of their successive differences, \(b_j = a_j - a_{j+1}\). In Paris three years earlier, while being guided into mathematical study by the Dutch mathematician, physicist and astronomer Christiaan Huygens (1629-1695), Leibniz had already noted that a difference sequence is easily summed, since it becomes what is now called a telescoping sum. Since for each \(j\) adding pairs of successive terms produces cancellations, i.e.

\[
    b_{j-1} + b_j = (a_{j-1} - a_j) + (a_j - a_{j+1}) = a_{j-1} - a_{j+1}
\]

it follows that for any \(n \geq 1\) we have

\[
    b_1 + b_2 + \ldots + b_n = a_1 - a_{n+1}.
\]

He had applied this when Huygens asked him to sum the reciprocals of triangular numbers 1, 3, 6, 10, 15, 21, ... Since a triangular number takes the form \(\frac{1}{2}k(k + 1)\), its reciprocal is \(\frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1}\). Taking \(a_k = \frac{1}{k}\), he obtained \(b_k = \frac{2}{k(k+1)}\). The partial sum \(b_1 + b_2 + \ldots + b_n\) of the first \(n\) terms of the series

\[
    \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \ldots
\]

became \(a_1 - a_{n+1} = 2 - \frac{2}{n+1}\). As \(\frac{2}{n+1}\) becomes infinitesimal for infinite \(n\), Leibniz concluded that the sum of \(b_1 + b_2 + \ldots + b_n + \ldots\) was 2.

Leibniz studied this and other examples in detail, coming to the realisation that the operations of forming difference sequences and sum sequences are in effect mutual inverses – each undoes the other. He applied this insight to geometric curves in the plane. Leibniz perceived a curve in the \((x, y)\)-plane as depicting the values taken by a variable quantity \(y\) whose changes in value depend on changes in the value of the variable \(x\). He considered the slope of the tangent to the curve (a straight line touching, but not crossing,
the curve) at various points in the \((x, y)\)-plane. He treated this slope as the ratio of infinitesimal increments \(dy\) and \(dx\) in \(y\) and \(x\) respectively. In Figure 28, he then takes each increment in \(x\) as the infinitesimal ‘unit’, \(dx_i = x_{i+1} - x_i\) (denoted here by 1), so that the tangent at the point \((x_i, y_i)\) becomes the difference \(dy_i = (y_{i+1} - y_i)\) of two successive ordinates.

The area under the curve, on the other hand, is taken as the sum of the areas of infinitely thin rectangles, with base vertices \(x_i\) and \(x_{i+1}\) on the \(x\)-axis and height given by the ordinate \(y_i\). Since the base of each rectangle is infinitesimal, he assumes that the heights along the top edge of each small rectangle remain infinitely close to the corresponding part of the graph of the curve. The base of each rectangle is one infinitesimal unit, so the area under the curve becomes the sum of the ordinates: \(y_1 + y_2 + \ldots + y_n\), where \(n\) is infinite.

Difference sequences and sum sequences are opposites. For Leibniz this illustrates the inverse relationship, for a given curve (denoted here by a function \(f\)), between differentiation, in which we find the values of its tangent curve (denoted by \(f'\)) at various points, and integration, where we seek to express \(f\) in terms of the area under the graph of its tangent curve \(f'\). The inverse relationship between these two basic operations became known as the Fundamental Theorem of the Calculus.

Leibniz’ results represented a major step forward in the use of the newfound algebraic symbolism to describe properties of curves and move beyond the confines of Greek geometry in the study of accelerated motion. However, his methodology led to serious foundational questions about the existence of the objects being studied, since it was by no means clear how the infinitesimal quantities could serve as fundamental building blocks on which a rigorous logical foundation of the Calculus, in the pattern of Euclid’s Elements, could ultimately be based.
He was fully aware that the use of infinitesimal quantities needed to be justified. He had no proof of the existence of such entities, but regarded them as ‘ideal quantities’, to be used in a formal framework for calculation. Using his *Principle of Continuity*, he asserted that these fictional ‘ideal’ numbers were governed by the same laws as ‘ordinary’ numbers, by which he meant rational or irrational numbers, the existence of the latter being justified by an appeal to the geometric number line.

At the same time, he and his followers made the infinitesimal increment $dx$ (called the *differential*) the basis of their calculations. They claimed that two quantities could be treated as the same if their difference was *infinitesimal* (smaller than any given positive quantity). The *instantaneous rate of change* in $y$ at the point $(x, y)$, which yields the slope of the tangent at the point, was then assumed to be given by the *ratio* $\frac{dy}{dx}$ of two infinitesimals.

The logical inconsistency of these claims was obvious – infinitesimals could not simultaneously be treated as zero and as non-zero quantities. Nevertheless, judicious and selective use of techniques for calculation based on the above premises provided an *operational* foundation for a highly successful Calculus with wide-ranging applications, which produced convincing answers to outstanding problems in mathematics and physics.

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8https://commons.wikimedia.org/wiki/File:SIR_Isaac_Newton_Mezzotint_by_J._Faber,_junior,_1726,_after_Wellcome_V0004265.jpg
3. INFINITESIMALS IN THE CALCULUS

3.3. Newton. Isaac Newton (1642-1727), whose development of the Calculus (but little of its publication) predates that of Leibniz by a decade or so, was more cautious in his description of infinitesimals.

To justify calculations that included infinitesimals he therefore relied primarily on his physical intuition and on ‘motion’. He discussed the distance covered by a moving ‘particle’ tracing out a curve over an infinitesimal time period. He described the ‘flow’, i.e. the change in position, of a variable \( x \) (the fluent) over an infinitesimal ‘instant’ \( \dot{o} \) by means of its velocity or fluxion, \( \dot{x} \). The change in position is then provided by the product \( \dot{x}o \).

Similarly, for a variable \( y \), whose values depend on those of \( x \), the change in position is \( \dot{y}o \). For example, if \( y = x^2 \), this yields
\[
\dot{y}o = (x + \dot{x}o)^2 - x^2 = 2x\dot{x}o + (\dot{x}o)^2.
\]

The relative velocity is the ratio of the two changes in position,
\[
\frac{\dot{y}o}{\dot{x}o} = 2x + \dot{x}o.
\]

Now Newton argues that \( \dot{o} \) is infinitesimal, so the final term can be neglected, and the ratio of the two fluxions is therefore given by \( 2x \). For each \( x \) in the abscissa (the \( x \)-axis), this ratio is then interpreted as the tangent to the curve \( y = x^2 \) at the point \( x \), and measures the curve’s instantaneous rate of change at this point. He uses this approach to analyse a wide range of curves. Clearly, unless the curve is a straight line, the slope of the tangent will vary as \( x \) varies.

Similarly, Newton computed the fluxion of \( y = x^3 \) by considering
\[
\dot{y}o = (x + \dot{x}o)^3 - x^3 = 3x^2\dot{x}o + 3x(\dot{x}o)^2 + (\dot{x}o)^3,
\]
so that the ratio in the changes of position becomes
\[
\frac{\dot{y}o}{\dot{x}o} = 3x^2 + 3x(\dot{x}o) + (\dot{x}o)^2,
\]
which he equates with \( 3x^2 \) as the last two terms again ‘vanish’.

To handle general integral powers \( y = x^n \), Newton used the binomial theorem, which (as above for \( n = 2, 3 \)) expresses \((a + b)^n\) as a finite sum of terms in \( a^k b^{n-k} \) for \( k = 0, 1, 2, ... n \). The binomial coefficient of \( a^k b^{n-k} \) takes the form
\[
\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!},
\]
where the denominator (\( k \) factorial) is given by the product \( k! = 1 \times 2 \times 3 \times ... \times k \). These coefficients can be read off from the rows of the famous ‘triangle’ of Blaise Pascal (1623-1662) shown in Figure 30, where each term is the sum of the two diagonally above it.

Newton applied the binomial theorem to find the fluxions of \( y = x^n \) just as in the examples we computed above, neglecting all terms that still include \( \dot{o} \) after division by \( \dot{x}o \). This ensures that the ratio of the fluxions (the relative rate of change in position) is \( \frac{\dot{y}}{\dot{x}} = nx^{n-1} \). This represents the instantaneous
rate of change in the variable \( y = x^n \) as \( x \) varies. Newton assumed (without proof) that the fluxion of the polynomial \( y = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \) can then be found term-by-term.

The method of fluxions was of critical importance in Newton’s physics, where, for a moving body, velocity (the rate of change in position) and acceleration (the rate of change in velocity) are calculated as fluxions, so that acceleration is the second-order fluxion of position (or distance travelled).\(^9\) Newton’s second law \( F = ma \) (force equals mass times acceleration) makes use of \( a \) as the second fluxion of position. Similarly one may repeat the process to seek the \( k^{th} \) fluxion for any whole number \( k \). In modern terminology this is called the \( k^{th} \) derivative, written by Leibniz as \( \frac{d^k y}{dx^k} \).

Finding the fluxion enabled Newton to compare the differences in position (the increments) of the \( x \) and \( y \) variables (hence differentiation). Going in the opposite direction turned out to be the same as Leibniz’ summation, i.e. integration. This required Newton to find the fluent \( y \) when its fluxion is known. He was clearly aware of the inverse relationship between these two processes. He explained this by example in his De Analysi (written in 1669 but not published until 1711).\(^{10}\) In his example he assumed the area under a certain unknown curve, taken from the origin up to some unspecified value \( x_0 \), to be given as \( \frac{2}{3} x_0^{\frac{3}{2}} \). He then reversed his perspective: considering \( z = \frac{2}{3} x^{\frac{3}{2}} \) as the fluent, he showed that its fluxion is \( y = x^{\frac{1}{2}} \), and that this is the curve for which the area under the graph was given as \( \frac{2}{3} x_0^{\frac{3}{2}} \). This demonstrated by example now the two operations may be seen as inverses of each other (see Figure 31).

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\(^9\)Velocity is the fluxion (rate of change) of distance, while acceleration is the fluxion of velocity, so that acceleration is the second-order fluxion of distance.

\(^{10}\)His method of fluxions was expounded in some detail in his Latin treatise Methodus fluxionem et serium infinitorum, written in 1671. This important, but technically difficult, work did not find a publisher in Newton’s lifetime, and first appeared in print in 1736 (in an English translation) and then in 1744 (retranslated) in Latin.
Newton extended his rules for finding fluxions to curves given by fractional powers of $x$, such as $x^{3/2}$. He showed, more generally, that the area under the graph of $y = x^{m/n}$ is given by

$$z = \left(\frac{m}{m + n}\right)x^{m/n}.$$

This was, in itself, a remarkable extension of the known results involving fluxions at that time. To achieve this, Newton extended the binomial theorem to fractional powers. This meant that $(a + b)^{m/n}$ could be expanded as an infinite series (a series with infinitely many terms), whose $n^{th}$ term took the form $a_nx^n$, where Newton found each constant coefficient $a_n$ by interpolation and analogy, rather than by a formal proof.\(^{11}\)

Examples like this led him to assert confidently in his 1669 treatise *De Analysi*:

> And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.

Newton employed series expansions of functions to great effect, enabling him to build extensive tables of fluxions (what we call derivatives) derived from given fluents. Conversely, he obtained the fluents from given fluxions (our integrals or antiderivatives), for a wide range of curves, always working term by term from the series expansion. His success encouraged others to work with series containing infinitely many terms containing increasing powers of $x$, in the same way as they had done with polynomials. They felt that the ‘power series’ so obtained could now safely be regarded as obeying the same algebraic laws as polynomials and provided an alternative representation of a host of different curves. Wallis (whose interpolation methods Newton had used extensively) expressed this conviction most clearly in his

\(^{11}\)More details, including a proof of the binomial theorem, can be found in *MM.*
Algebra, arguing that infinite series ‘intimate the designation of some particular quantity by a regular Progression or rank of quantities, continually approaching to it; and which, if infinitely continued, must be equal to it’.

Concerns over the validity of the methods employed in the Calculus took some time to emerge, perhaps obscured by the evident success of Newton’s and Leibniz’ results in solving outstanding problems. A more immediate question concerned the meaning of infinitesimals in the conceptual frameworks that Cavalieri, Leibniz and Newton had employed. Some of these concerns will be outlined in Section 4.

Insistence on motion remains present in many of Newton’s mathematical writings, and he uses it in his attempt to justify his fluxional calculus. He argues that mathematical quantities are not to be seen as ‘composed of Parts extremely small, but as generated by a continual motion’.

In his famous *Principia Mathematica* (1687) he states: ‘Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of time approach nearer to each other than by any given difference $D$, become ultimately equal.’ (This is, in effect, his version of the Principle of Continuity.)

Finally, in *Tractatus de quadratura curvarum* (1693) he argues that ‘fluxions are very nearly the Augments of the Fluents, generated in equal, but infinitely small parts of Time, and to speak exactly, are in the Prime Ratio of the nascent Augments.....’Tis the same thing if the Fluxions be taken in the Ultimate Ratio of the Evanescent Parts’.

However, recourse to infinitely divisible time, rather than space, does not offer a way out of the dilemma. Newton’s calculations clearly violate another key principle that he had stated as: ‘...Errors, tho’ never so small, are not to be neglected in Mathematics’. His instant $o$ cannot be used when dividing the ‘instantaneous’ change in $y$ by the corresponding change in $x$ and then be ‘neglected’ in the same breath!

3.4. Euler and the natural logarithm. Nevertheless, development of the methods of the Calculus and its many applications led to an increasingly sophisticated quantitative analysis of all forms of dynamics throughout the eighteenth century, with most practitioners paying scant attention to the underlying inconsistencies in its mathematical foundations. The prolific and highly influential Swiss mathematician Leonhard Euler (1707-1783), writing in the 1740s, made clear that he regarded the use of infinitesimals as the basic tool in handling differentiation, describing it as ‘a method for determining the ratios of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment’.$^{12}$

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$^{12}$Euler had severe problems with his eyesight from at least 1738, and became completely blind in the 1760s. Despite this handicap, he produced over 800 books and articles, ranging over
An important innovation in his writings was that the concept of function (usually given by a formula) replaced that of a curve as the basis of his analysis – thereby moving it away from visual representations while greatly widening its scope. In his ground-breaking 1748 treatise Introductio in analysin infinitorum (Introduction to analysis of the infinite) Euler says: ‘A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.’ Although this definition was later modified, it marked a significant shift away from reliance on curves and geometric representation.13

In his Introductio he displayed great (and only occasionally unfounded) confidence in dealing with equations with infinitely many terms, and with power series in particular. He also used infinitesimals, as well as their reciprocals, infinite ‘numbers’, freely. His genius lay in (usually) arriving at correct results, even though the methods he used often could not be justified rigorously.

In particular, Euler set out to give definitions of key classes of function, such as polynomials, exponentials (which for him were ‘simply powers whose exponents are variable’) and logarithms (which were the ‘inverse of these’). His derivation of the natural logarithm provides a good case study in mid-eighteenth century Calculus.

John Napier, when developing his logarithmic tables (see Chapter 3), had correctly appreciated the usefulness of using a geometric progression with common ratio $s_{n+1} = s_n$ very close to 1, but unfortunately insisted on using a decreasing sequence, as he wished to keep his ‘whole sine’ sufficiently large.

The proof that, as $n$ grows ever larger, the numbers $s_n = (1 + \frac{1}{n})^n$ will in fact settle down to a definite value (lying between 2 and 3) is credited to Jacob Bernoulli, who had been investigating growth rates of investments accruing at compound interest rates, with compounding happening at ever shorter time intervals. At that stage, however, no-one had yet associated the limiting value with logarithms.

It was Euler who first considered the elusive ‘limit’ $e$ of the increasing sequence $(s_n)_{n \geq 1}$ with $s_n = (1 + \frac{1}{n})^n$ as the base of a system of logarithms. In Chapter 6 of his Introductio, Euler discussed exponents and logarithms and also applied his results to problems such as population growth rates and the amortisation period of a loan attracting periodic compound interest.

In Chapter 7 he considered logarithms relative to a general base $a > 1$, and noted that, if $\omega$ is an ‘infinitely small quantity’, he can write $a^{\omega} = 1 + \psi$, where $\psi$ is also infinitely small. He assumed that his choices would allow

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13The development of the modern concept of function is outlined in MM.
him to write $\psi = k\omega$ for some finite $k$, so that $a^\omega = 1 + k\omega$, hence

$$\omega = \log_a (1 + k\omega).$$

Taking an ‘infinite number’ $j$, he then expressed $a^{\omega j} = (a^\omega)^j = (1 + k\omega)^j$ as a power series.

To do this, Euler applied the binomial theorem directly to the sum of 1 and the infinitely small number $k\omega$, as well as using the infinite power $j$, writing

$$(1 + k\omega)^j = 1 + \frac{j}{1!}k\omega + \frac{j(j-1)}{2!}k^2\omega^2 + \frac{j(j-1)(j-2)}{3!}k^3\omega^3 + ...$$

Next, he again supposed that $j = \frac{z}{\omega}$ for some finite $z$, so that $z = \omega j$ and $\omega = \frac{z}{j}$.

Since $a^z = (a^\omega)^j = (1 + k\omega)^j$, the series expansion now read

$$a^z = 1 + \frac{1}{1!}kz + \frac{j(j-1)}{2!}k^2\frac{(z)^2}{j^2} + \frac{j(j-1)(j-2)}{3!}k^3\frac{(z)^3}{j^3} + ...$$

$$= 1 + \frac{1}{1!}kz + \left(\frac{j-1}{j}\right)\frac{1}{2!}k^2z^2 + \left(\frac{(j-1)(j-2)}{j^2}\right)\frac{1}{3!}k^3z^3 + ...$$

But since $j$ is infinite, he argued that the ratios $\frac{j-1}{j}, \frac{(j-1)(j-2)}{j^2}$, etc., would all cancel (!!), leaving him with the expansion

$$a^z = 1 + \frac{kz}{1!} + \frac{k^2z^2}{2!} + \frac{k^3z^3}{3!} + ...$$

with all three of $a, z, k$ as finite numbers.

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14https://commons.wikimedia.org/wiki/File:Leonhard_Euler_by_Darbes.jpg
The simplest case was $k = 1$. Euler now reserved the symbol $e$ for the value of $a$ in that case, deriving a formula that has become a staple of modern mathematics:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.$$ 

Having taken $k = 1$, the relation $a^{\omega j} = (1 + k \omega)^j$ that he started with now read $e^z = (1 + \omega)^j = (1 + \frac{j}{j})^j$ for this ‘infinite’ value of $j$. Taking $z = 1$, he treated $(1 + \frac{1}{j})^j$ as the limiting value of the sequence $s_n = (1 + \frac{1}{n})^n$ when $n$ grows indefinitely. He had therefore found the limiting value (as $n$ grows) of the $s_n$ to be the number

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$ 

Euler proved that $e$ is irrational. He calculated its decimal expansion to 23 decimal places, and proceeded to use $e$ as the base for what we now call natural logarithms.\(^{15}\) When $x = e^y$, we write $y = \log_e x$.

In order to approximate the natural logarithm of a given positive number $x$, Euler used calculations similar to the above (again using infinitesimal and infinite numbers freely) to derive infinite series expansions for $\log_e(1 + x)$ and $\log_e(1 - x)$, arriving at the following series, from whose partial sums such logarithmic tables could be established:

$$\log_e \left( \frac{1 + x}{1 - x} \right) = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n-1}}{2n-1} + \ldots)$$

He was not yet done. Leibniz and Johann Bernoulli had expressed conflicting views on the nature of $\log_e(-x)$ and their discussion led Euler to consider how to extend the logarithmic function to negative numbers. Bernoulli had argued that $\log_e(-x)$ should equal $\log_e(x)$, since both yield the derivative $\frac{1}{x}$; while Leibniz argued that the rule $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ assumed that $x > 0$. Euler pointed out that two functions that differ by a constant have equal derivatives, so that one cannot conclude that the functions themselves will be equal if their derivatives are equal.

By definition of the logarithm, $\log_e(-x) = \log_e((-1) \times x) = \log_e(x) + \log_e(-1)$, as Euler pointed out. To determine the value of the final term on the right, he would use the familiar de Moivre formulae, which, in his hands, became a fundamental tool in complex analysis.\(^{16}\) Writing $i$ for the ‘imaginary unit’ $\sqrt{-1}$ (this became the standard notation) he knew that, for any

\(^{15}\)The reason for calling logarithms to this base natural relates to its definition, which can be given (as above) as the inverse of the exponential function, or as the integral of the function $g(x) = \frac{1}{x}$. The exponential function $f(x) = e^x$ has the unique property that it equals its derivative: $f'(x) = e^x$, so that, at any point $x$, its instantaneous rate of growth is equal to its value. This concept (and its generalisations) has many applications to models of population growth, continuous compounding, etc. See [6], and see MM for a proof that the number $e$ is irrational.

\(^{16}\)See MM for the derivation of these formulae.
\( n \geq 1, \)
\[
\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n, \quad \cos(n\theta) - i \sin(n\theta) = (\cos \theta - i \sin \theta)^n.
\]

Adding provides the identity \( \cos(n\theta) = \frac{(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n}{2}, \) and, setting \( x = n\theta \) and taking \( n \) ‘infinite’ (so that \( \theta = \frac{x}{n} \) is infinitesimal), Euler deduced that \( \cos(\frac{x}{n}) = 1 \) and \( \sin(\frac{x}{n}) = \frac{x}{n}. \) Substituting this into the above he obtained,
\[
\cos x = \frac{(1 + i\frac{x}{n})^n + (1 - i\frac{x}{n})^n}{2} = \frac{e^{ix} + e^{-ix}}{2},
\]
where the final identity follows because, when \( n \) is infinite and \( z \) finite, we have \( e^z = (1 + \frac{z}{n})^n \) as noted above, and can apply this to the finite (imaginary) quantity \( z = ix. \) An exactly analogous argument with \( i \sin(n\theta) \) shows that \( i \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \) so that he derived what we now call Euler’s identity:
\[
e^{ix} = \cos x + i \sin x.
\]

He had shown that complex exponentials could, as he put it, ‘be expressed by real sines and cosines’.

A celebrated identity arises when we take \( x = \pi : \) we have \( e^{i\pi} = -1, \) i.e.
\[
e^{i\pi} + 1 = 0.
\]

This, it is sometimes argued, links the five ‘most important’ numbers in mathematics: 0, 1, \( e, \pi \) and \( i. \) It may also have served to persuade many observers that the ‘mysterious’ square root of \(-1 \) needed to be understood more fully.

Finally, taking the natural logarithm on both sides of the identity \( e^{i\pi} = -1, \) Euler noted that \( \log_e(-1) = i\pi. \) Thus the logarithm of a negative number is purely imaginary. Euler went on to deduce (correctly), that the logarithm of a complex number is not single-valued, but has infinitely many branches – but we will leave the matter there.

4. Critique of the calculus

In Britain, serious questions about the foundations of the Calculus were raised publicly soon after Newton’s death in 1727, in a way that could not easily be ignored. The fundamental inconsistency of early Calculus techniques was seized upon by the philosopher and cleric George Berkeley (1685-1753), Bishop of Cloyne in Ireland. Berkeley’s explicit purpose was to defend religious faith against assertions by the Astronomer Royal Edmund Halley and others (although not Newton himself) that scientific and mathematical progress had rendered faith in scriptural revelation redundant. \(^{17}\) In his 1734 tract The Analyst (whose subtitle begins: A DISCOURSE addressed to an infidel MATHEMATICIAN...) Berkeley examines whether the new Calculus really was as soundly based as had been claimed; or as he put it:

\(^{17}\) Halley had mocked the earlier tract Alciphron by Berkeley; it is also claimed that Halley had persuaded a friend of Berkeley’s to renounce religion on his deathbed.
Whether such mathematicians as cry out against mysteries have ever examined their own principles?

(The Analyst, Question 63)

Berkeley argued, correctly, that the Calculus of Newton and Leibniz rested on the use of infinitely small quantities whose existence was unproven and logically dubious. He pointed out that such infinitesimals were treated as actual (non-zero) quantities in calculations, yet were later declared to be negligible. Fastening on Newton’s notion of ultimate ratios, he asked memorably:

"And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"

His critique, arguing that mathematicians are as reliant on faith as theologians, and worked by ‘submitting to authority, taking things on trust’, hit home in British mathematical circles, and provided motivation for various attempts by prominent mathematicians to improve the foundations of the Calculus.\(^{18}\) Of these responses perhaps the most complete was *Treatise on fluxions* (1742) by the Scottish mathematician Colin Maclaurin (1698-1746), a major two-volume work which set the Calculus in a geometric framework and further developed the theory of power series, but did not really rebut Berkeley’s critique.

Nonetheless, as we have seen, and just as proved to be the case with the discovery of incommensurables like \(\sqrt{2}\) more than two millennia earlier, a lack of proper foundations for their new methods of analysis in no way delayed most mathematicians (especially on the Continent) in their development of the Calculus and its many applications in the natural sciences throughout the seventeenth and eighteenth centuries.

It continued to cause concern to philosophers, however, and even troubled the eminent empiricist David Hume (1711-1776). On the one hand, Hume famously ends his *Enquiry into Human Understanding*\(^{22}\) with a clarion call to his readers:

*If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask; Does it contain any abstract reasoning concerning quantity or number? No. Does it contain any experimental reasoning concerning matter of fact and existence? No. Commit it then to the flames; for it can contain nothing but sophistry and illusion.*

However, despite his high regard for mathematical reasoning, just a few pages earlier in the same volume Hume expresses with great clarity

\(^{18}\) Berkeley also questioned whether irrationals, such as the diagonal of the unit square, should be treated as numbers. This aroused rather less concern at the time.
his acute anxiety over the meaning of an apparent hierarchy of infinitely divisible quantities:

The chief objection against all abstract reasonings is derived from the ideas of space and time; ideas, which, in common life and to a careless view, are very clear and intelligible, but when they pass through the scrutiny of the profound sciences (and they are the chief object of these sciences) afford principles, which seem full of absurdity and contradiction. No priestly dogmas, invented on purpose to tame and subdue the rebellious reason of mankind, ever shocked common sense more than the doctrine of the infinite divisibility of extension, with its consequences; as they are pompously displayed by all geometricians and metaphysicians, with a kind of triumph and exultation. A real quantity, infinitely less than any finite quantity, containing quantities infinitely less than itself, and so on in infinitum; this is an edifice so bold and prodigious, that it is too weighty for any pretended demonstration to support, because it shocks the clearest and most natural principles of human reason.

Hume’s particular example, which, as he points out, relies only on ‘the clearest and most natural’ chain of reasoning (properties of circles and triangles), is the notion of the angle of contact, or horn angle, between a straight line and a curve (such as that between the circumference of a circle and its tangent). This angle is shown to be ‘infinitely less than any rectilineal angle’ and can be made ever smaller simply by increasing the diameter of the circle. While the proof of this claim seems ‘as unexceptionable as that which proves the three angles of a triangle to be equal to two right angles’, it palpably offends common sense, in his view. Hume could equally well have said the same about perceptions that the number line can contain similarly incomparable quantities.