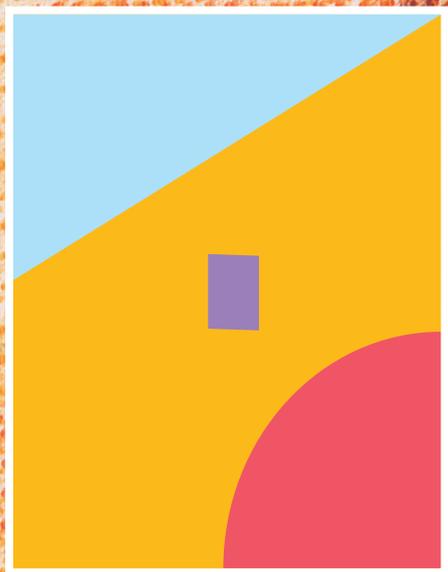


Models in Microeconomic Theory

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8 A jungle

In this chapter and the next we study a society consisting of a set of individuals and a set of houses. Each house can accommodate only one person and each person can occupy only one house. Different people may have different preferences over the houses, but everyone prefers to occupy any house than to be homeless.

In this chapter we analyze a model in which the assignment of houses is determined by the individuals' strengths; the concepts of property and ownership do not exist. A person who wants to occupy a house currently occupied by a weaker person can do so simply by presenting herself to the current occupant. The process is orderly: everyone knows everyone else's strength, and on seeing that a stronger person wants to occupy her house, a person vacates it without a fight, which she knows she would lose.

We study the existence and character of a stable assignment of houses to individuals. Will the people forever be evicting each other, or does an assignment of people to houses exist in which no one wants the house of anyone who is weaker than her? What are the properties of such an assignment?

8.1 Model

Society

A society is defined by a set of individuals, a set of houses, and the individuals' preferences over the houses. To simplify the analysis, we assume that the number of houses is equal to the number of individuals and that no individual is indifferent between any two houses.

Definition 8.1: Society

A *society* $\langle N, H, (\succ^i)_{i \in N} \rangle$ consists of

individuals

a finite set N

houses

a finite set H with the same number of members as N

preferences

for each individual $i \in N$, a **strict preference relation** \succ^i over H .

We interpret $h \succ^i h'$ to mean that individual i prefers to occupy house h than house h' . Notice that we assume that each individual cares only about the house she occupies, not about the house anyone else occupies. We discuss this assumption in [Section 8.6](#).

An outcome of the model is an assignment of houses to individuals. Formally, an assignment is a function from the set N of individuals to the set H of houses. We typically denote an assignment by a , with $a(i)$ being the house assigned to individual i . Since we assume that a house can be occupied by at most one individual, an assignment is feasible only if it assigns different individuals to different houses (that is, only if it is a one-to-one function). We call such an assignment an allocation.

Definition 8.2: Assignment and allocation

An *assignment* for a **society** $\langle N, H, (\succ^i)_{i \in N} \rangle$ is a function from the set N of individuals to the set H of houses, associating a house with every individual. An *allocation* is an assignment in which each house is assigned to exactly one individual.

Example 8.1

The following table gives an example of a society with four individuals, 1, 2, 3, and 4, and four houses, A , B , C , and D . Each column indicates the preference ordering of an individual, with the individual's favorite house at the top. For example, individual 1's preference ordering is $B \succ^1 C \succ^1 D \succ^1 A$.

Individuals				
1	2	3	4	
B	B	A	B	
C	D	B	C	
D	A	C	A	
A	C	D	D	

One allocation, in which individuals 1, 2, 3, and 4 occupy houses D , A , C , and B respectively, is highlighted.

Power

What determines the allocation of houses? Presumably the individuals' preferences play a role. In this chapter, we focus on an additional factor: the individuals' relative power. We assume that between any two individuals there is a stable power relation: either i is stronger than j , or j is stronger than i . No two individuals are equally powerful. Precisely, we take as given a binary relation \triangleright on the set of individuals, where $i \triangleright j$ means that i is more powerful than j , in which case i can take over the house occupied by j .

Definition 8.3: Jungle

A *jungle* $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$ consists of a *society* $\langle N, H, (\succ^i)_{i \in N} \rangle$ and a *power relation* \triangleright , a **complete, transitive, antisymmetric** binary relation on the set N of individuals.

Comments

1. *Power is not only physical.* One individual may be more powerful than another because she is physically stronger. But alternatively, her social status, ability to persuade, or seniority may be sufficient to allow her to force another individual to relinquish the house that individual occupies.
2. *No property rights.* The model makes no reference to property rights. An individual can occupy a house, but does not *own* it. No legal system by which an individual can defend her occupation of a house exists. In many societies, property rights interfere, to some extent, with the exercise of power. Many models in economics consider the extreme case in which property rights are perfectly enforced. In this chapter, we consider the other extreme, where power alone determines the outcome.
3. *Outcome of a struggle is deterministic.* The notion of power is deterministic: if $i \triangleright j$, then in a contest between i and j , i wins for sure; there is no chance that j wins.
4. *No cost of fighting.* Vacating a house involves no cost. Whenever the occupant of a house is confronted by a stronger individual, the occupant recognizes her inferiority and costlessly vacates her house.
5. *No coalitions.* The model specifies the actions possible for each individual; it does not include any separate specification of the actions possible for groups of individuals. This formulation reflects an implicit assumption that each individual acts on her own; no group can achieve any outcome that its members cannot achieve by themselves.

8.2 Equilibrium

An equilibrium in a model is generally an outcome that is stable given the forces assumed to be active. In a **jungle**, a stable outcome is an **allocation** in which no individual prefers any house that she can obtain to the one she occupies. Thus an allocation is *not* an equilibrium if some individual prefers the house occupied by a weaker individual to the house she currently occupies.

More precisely, an equilibrium of a jungle is an allocation a for which no individual i prefers the house $a(j)$ occupied by any individual j who is weaker than i to the house $a(i)$ that she occupies.

Definition 8.4: Equilibrium

An *equilibrium* of the jungle $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$ is an **allocation** a^* such that for no individuals $i, j \in N$ is it the case that $i \triangleright j$ and $a^*(j) \succ^i a^*(i)$.

Example 8.2

For the jungle consisting of the society in **Example 8.1** together with the power relation \triangleright for which $1 \triangleright 2 \triangleright 3 \triangleright 4$, the allocation (B, D, A, C) , highlighted in the following table, is an equilibrium.

Individuals				
1	2	3	4	
B	B	A	B	
C	D	B	C	
D	A	C	A	
A	C	D	D	

Individuals 1 and 3 occupy their favorite houses. Individuals 2 and 4 do not occupy their favorite house, which is occupied by a stronger individual. Individual 2 prefers D to the houses occupied by the individuals weaker than her, namely 3 and 4; no individual is weaker than 4.

Every jungle has an equilibrium. That is, for every set of individuals, set of houses, profile of preference relations, and power relation, an allocation—at least one, possibly more than one—is stable against the exercise of power.

We prove this result by showing that an allocation generated by a procedure called *serial dictatorship* is an equilibrium. This procedure is defined for an arbitrary ordering of the individuals. It assigns to the first individual in the ordering her favorite house, say h_1 , and to the second individual in the ordering her favorite house among all those that remain after removing h_1 . It continues in the

same way according to the ordering of the individuals, assigning to each individual her favorite house among all houses that remain after removing the ones assigned to the individuals who precede her in the ordering.

Procedure: Serial dictatorship

For the **society** $\langle N, H, (\succ^i)_{i \in N} \rangle$ and ordering i_1, i_2, \dots, i_n of the members of N , the *serial dictatorship procedure* generates the allocation defined inductively as follows.

Initialization

The house h_1 allocated to individual i_1 is the best house in H according to i_1 's preference relation, \succ^{i_1} .

Inductive step

For every $k \geq 2$, the house h_k allocated to individual i_k is the best house in $H \setminus \{h_1, \dots, h_{k-1}\}$ according to i_k 's preference relation, \succ^{i_k} .

If we apply this procedure to the society in [Example 8.1](#) and the ordering 1, 2, 3, 4 then we get the allocation (B, D, A, C) , which [Example 8.2](#) shows is an equilibrium of the jungle with the power relation \triangleright for which $1 \triangleright 2 \triangleright 3 \triangleright 4$. We now show that this result is general.

Proposition 8.1: Every jungle has an equilibrium

For any **jungle** $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$, the **allocation** generated by the **serial dictatorship procedure** for the **society** $\langle N, H, (\succ^i)_{i \in N} \rangle$ and the ordering \triangleright is an **equilibrium**.

Proof

Let a be the assignment generated by the serial dictatorship procedure for the society $\langle N, H, (\succ^i)_{i \in N} \rangle$ and the ordering \triangleright . The number of houses is the same as the number of individuals, so the procedure assigns to every individual a house. Every house is assigned only once, so the assignment a is feasible, and hence an allocation. The allocation is an equilibrium because the house assigned to each individual is the best house, according to the individual's preferences, among all the houses not assigned to stronger individuals.

Note that the serial dictatorship procedure is not the equilibrium concept; it is only a means by which to prove that an equilibrium exists. [Proposition 8.1](#) leaves open the possibility that other equilibria, constructed differently, exist, but we now show that in fact no other equilibria exist.

Proposition 8.2: Every jungle has a unique equilibrium

Every jungle has a unique equilibrium.

Proof

Consider the jungle $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$. Without loss of generality assume that $N = \{1, \dots, n\}$ and $1 \triangleright 2 \triangleright \dots \triangleright n$. Assume, contrary to the claim, that a and b are two equilibria of the jungle. Denote by i^* the strongest individual i for whom $a(i) \neq b(i)$. That is, $a(i) = b(i)$ for all $i < i^*$ and $a(i^*) \neq b(i^*)$, as illustrated in the following diagram.

$i:$	1	2	...	$i^* - 1$	i^*	...	j	...	k	...	n
$a(i):$	■	■	...	■	■	...	■	...	■	...	■
$b(i):$	■	■	...	■	■	...	■	...	■	...	■

The set of houses allocated to individuals 1 through $i^* - 1$ is the same in a and b , so the set of houses allocated to i^* through n is also the same in the two allocations. Thus the house $a(i^*)$ (orange in the diagram) is allocated by b to some individual j less powerful than i^* : $a(i^*) = b(j)$. Also the house $b(i^*)$ (blue) is allocated by a to some individual k less powerful than i^* : $b(i^*) = a(k)$. (Individual j could be more or less powerful than k ; in the diagram she is more powerful.) The fact that a is an equilibrium implies that $a(i^*) \succ_{i^*} a(k) = b(i^*)$ and the fact that b is an equilibrium implies that $b(i^*) \succ_{i^*} b(j) = a(i^*)$, a contradiction.

Comments

1. *Equilibrium is static.* The concept of equilibrium in a jungle, like the other equilibrium concepts in this part of the book, is static. Problem 5 describes a dynamic process that starts from an arbitrary initial allocation and asks you to show that this process converges to an equilibrium.
2. *Strict preferences.* The assumption that the individuals' preferences are strict (that is, no individual is indifferent between any two houses), is essential for this result. Problem 1 asks you to show that if one or more individuals are indifferent between two houses then a jungle may have more than one equilibrium.

8.3 Pareto stability

Now suppose that the allocation of houses to people is determined not by the balance of power, but by mutual agreement. One allocation is replaced by

another only if everyone agrees: no one objects and at least one person prefers the new allocation. Thus an allocation is immune to replacement if no allocation is better for some people and no worse for anyone.

Formally, an allocation b Pareto dominates an allocation a if for every individual the house assigned by b is at least as good as the house assigned by a , and at least one individual prefers the house assigned by b to the one assigned by a . An allocation is Pareto stable if no allocation Pareto dominates it.

Definition 8.5: Pareto stability

The allocation b in the society $\langle N, H, (\succ^i)_{i \in N} \rangle$ Pareto dominates the allocation a if $b(i) \succ^i a(i)$ for all $i \in N$ and $b(i) \succ^i a(i)$ for some $i \in N$. An allocation is Pareto stable if no allocation Pareto dominates it.

Consider the allocation in Example 8.1; denote it by a . Let b be the allocation in which $b(1) = C$, $b(2) = D$, $b(3) = A$, and $b(4) = B$. Individuals 1, 2, and 3 prefer the house assigned to them by b to the one assigned to them by a , and individual 4 occupies the same house in both allocations. Thus b Pareto dominates a and hence a is not Pareto stable.

Note that under the assumption that no individual is indifferent between any two houses, an allocation a is Pareto stable if for no allocation b does every individual who is allocated different houses by a and b prefer the house she is allocated by b to the one she is allocated by a .

Comments

1. *Terminology.* You may have previously encountered the term “Pareto optimal” or “Pareto efficient”. These terms are different names for “Pareto stable”. We use this terminology for two reasons. First, we want to emphasize that Pareto stability is an equilibrium concept. The force that can upset an allocation is an agreement between all individuals to replace the allocation with another one. Second, we want to use a name that has no normative flavor. A Pareto stable allocation might be good or bad, fair or unfair. Making such assessments requires information not present in the model.
2. *The notion of Pareto stability does not involve power.* Whether an allocation is Pareto stable depends only on the characteristics of the society, not on the power relation.
3. *Another potential source of instability.* Behind the definition of Pareto stability lies the assumption that whenever one allocation is Pareto dominated by another, the latter will replace the former. This assumption is strong. If the allocations b and c both Pareto dominate a , and b is better than c for some

individuals whereas c is better than b for others, then the individuals might disagree about the allocation that should replace a . The concept of Pareto stability implicitly assumes that this disagreement is not an obstacle to the replacement of a .

We now show that for any society and *any* ordering of the individuals, the allocation generated by the **serial dictatorship procedure** is Pareto stable.

Proposition 8.3: Serial dictatorship allocation is Pareto stable

For any **society** and any ordering of the individuals, the **allocation** generated by the **serial dictatorship procedure** is **Pareto stable**.

Proof

Let $\langle N, H, (\succ^i)_{i \in N} \rangle$ be a society and i_1, i_2, \dots, i_n an ordering of the individuals in N . Denote by a the allocation generated by the serial dictatorship procedure for the society with this ordering. That is, $a(i_1)$ is i_1 's most preferred house, $a(i_2)$ is i_2 's most preferred house in $H \setminus \{a(i_1)\}$, and so on.

Suppose, contrary to the claim that a is Pareto stable, that b is an allocation with $b(i) \succ^i a(i)$ for every individual $i \in N$ and $b(i) \succ^i a(i)$ for at least one individual $i \in N$. Let i_r be the first individual in the ordering i_1, i_2, \dots, i_n for whom $b(i_r) \succ^{i_r} a(i_r)$. Then $b(i_q) = a(i_q)$ for every $q < r$ (because no individual is indifferent between any two houses), as illustrated in the following diagram.

$i:$	i_1	i_2	...	i_{r-1}	i_r	...	i_s	...	i_n
$a(i):$	▲	▲	...	▲	▲	...	▲	...	▲
$b(i):$	▲	▲	...	▲	▲	...	▲	...	▲

Therefore $b(i_r) = a(i_s)$ for some individual i_s with $s > r$, and hence $a(i_s) \succ^{i_r} a(i_r)$, contradicting the fact that $a(i_r)$ is i_r 's favorite house among the houses that remain after removing those allocated to individuals who precede her in the ordering.

8.4 Equilibrium and Pareto stability in a jungle

We now connect the notion of **equilibrium**, which requires that an outcome be immune to the use of power, to the concept of **Pareto stability**, which requires that the outcome be immune to a reallocation to which everyone agrees.

Proposition 8.4: Equilibrium of every jungle is Pareto stable

The equilibrium of every jungle is Pareto stable.

Proof

By Proposition 8.2 every jungle has a unique equilibrium, and by Proposition 8.1 this equilibrium is obtained by applying the serial dictatorship procedure for the power relation. Thus by Proposition 8.3 the equilibrium is Pareto stable.

In Section 8.6 we show that this result depends on the assumption that each individual cares only about the house she occupies: equilibria of some jungles in which individuals care about the houses occupied by other individuals are not Pareto stable.

8.5 Which allocations can be obtained by a social planner who controls the power relation?

Now consider a social planner who can determine the power relation in the society but cannot dictate the allocation of houses. We assume that the planner believes that whatever power relation she dictates, the outcome will be an equilibrium of the jungle for that power relation. Which allocations can the planner induce? More precisely, for which allocations a is there a power relation such that a is an equilibrium of the jungle with that power relation? By Proposition 8.4 all equilibria are Pareto stable, so a necessary condition for an allocation to be achievable as an equilibrium for some power relation is that it be Pareto stable. We now show that this condition is in fact also sufficient. That is, a planner can, by choosing the power relation appropriately, induce any Pareto stable allocation. To prove this result we use the following lemma.

Lemma 8.1: Pareto stable allocation and favorite houses

In every Pareto stable allocation of any society, at least one individual is allocated her favorite house.

Proof

Let a be an allocation in the society $\langle N, H, (\succ^i)_{i \in N} \rangle$ for which $a(i) \neq h^*(i)$ for every $i \in N$, where $h^*(i)$ is i 's favorite house. We show that a is not Pareto stable.

Choose, arbitrarily, individual i_1 . By our assumption, $a(i_1) \neq h^*(i_1)$: i_1 does not occupy her favorite house. So that house, $h^*(i_1)$, is occupied by some other individual, say i_2 : $a(i_2) = h^*(i_1)$.

Similarly, i_2 's favorite house, $h^*(i_2)$, is occupied by an individual other than i_2 , say i_3 : $a(i_3) = h^*(i_2)$. This individual could be i_1 or some other individual.

If $i_3 = i_1$, define the allocation b by $b(i_1) = a(i_2) = h^*(i_1)$, $b(i_2) = a(i_1) = h^*(i_2)$, and $b(j) = a(j)$ for every other individual. That is, switch the houses of i_1 and i_2 and keep everyone else's house the same. Then both i_1 and i_2 prefer the house allocated to them in b to the house allocated to them in a , and everyone else occupies the same house in both allocations. Thus a is not Pareto stable.

If $i_3 \neq i_1$, continue in the same way to construct a sequence of individuals such that for each k , i_k 's favorite house, $h^*(i_k)$, is $a(i_{k+1})$, the one allocated by a to i_{k+1} . Because the set of houses is finite, for some $k \leq n$ we have $i_k = i_m$ for some $m < k$.

The following diagram illustrates the construction. An arrow from i to j means that i 's favorite house is the one allocated by a to j .

$$i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_m \rightarrow i_{m+1} \rightarrow \dots \rightarrow i_k$$

Now define the allocation b by $b(i_j) = a(i_{j+1})$ for all $j = m, \dots, k-1$, $b(i_k) = a(i_m)$, and $b(i) = a(i)$ for every other i . Every individual i_j for $j = m, \dots, k$ prefers the house allocated to her in b to the one allocated to her in a and every other individual occupies the same house in both allocations. Thus a is not Pareto stable.

We now show that by choosing the power relation appropriately, a social planner can achieve any Pareto stable allocation as the unique equilibrium.

Proposition 8.5: Pareto stable allocation is equilibrium for some power relation

Let a be a Pareto stable allocation of a society $\langle N, H, (\succ^i)_{i \in N} \rangle$. There exists a power relation \triangleright such that a is the only equilibrium of the jungle $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$.

Proof

We construct the power relation \triangleright as follows.

By Lemma 8.1, at least one individual is allocated her favorite house by

a. Let i_1 be such an individual, and start constructing \triangleright by making i_1 the most powerful individual.

Now remove i_1 from the set of individuals and $a(i_1)$ from the set of houses. That is, consider the society in which the set of individuals is $N \setminus \{i_1\}$ and the set of houses is $H \setminus \{a(i_1)\}$. The allocation a restricted to the individuals in this smaller society—that is, the allocation a' defined by $a'(i) = a(i)$ for all $i \in N \setminus \{i_1\}$ —is Pareto stable in the smaller society. The reason is that if it were not, there would be an allocation b' in the smaller society for which some individual in the smaller society is better off than she is in a' , and no individual is worse off than she is in a' . But then in the allocation b in the original society defined by $b(i_1) = a(i_1)$ and $b(i) = b'(i)$ for all $i \in N \setminus \{i_1\}$ Pareto dominates a , contradicting the Pareto stability of a in the original society.

Given the Pareto stability of a' in the smaller society, again by [Lemma 8.1](#) there exists an individual, say i_2 , who is allocated her favorite house in $H \setminus \{a(i_1)\}$ by a' . We continue the construction of \triangleright by making her the second most powerful individual.

Continue in the same way. At stage $k + 1$, identify an individual i_{k+1} for whom $a(i_{k+1})$ is the favorite house among $H \setminus \{a(i_1), \dots, a(i_k)\}$ and make her the $(k + 1)$ th most powerful individual.

By construction, for any individual i the house $a(i)$ allocated to her is better according to her preferences than every house allocated to an individual who is weaker according to \triangleright . Thus a is an equilibrium, and hence by [Proposition 8.2](#) the only equilibrium, of the jungle $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$.

Comment

Assume that you observe a group of individuals occupying a set of houses. You know the preferences of each individual (so that you know the society) but you do not know the power relation in the group. You want to determine whether the allocation you observe is consistent with an equilibrium of a jungle. To do so, you need to check whether there is a power relation such that the allocation you observe is the equilibrium of the jungle consisting of the society accompanied by that power relation.

By [Proposition 8.4](#) an allocation that is not Pareto stable is not an equilibrium, so that its appearance is inconsistent with equilibrium. On the other hand, by [Proposition 8.5](#) every Pareto stable allocation *is* an equilibrium of some jungle. So the observation of an allocation over which individuals do not quarrel is consistent with the allocation's being the outcome of the power struggle captured by a jungle.

8.6 Externalities

So far we have assumed that each individual cares only about the house she occupies. In fact, people may care not only about the houses they occupy but also about the allocation of houses to other people. For example, people generally care about their neighbors. They may care also about the appropriateness of other people's houses relative to their needs. They may consider, in addition, the fairness of the allocation. For example, a person may prefer an allocation in which everyone gets her second best house to an allocation in which she gets her favorite house but others get houses they rank near the bottom of their preferences.

The influence of one person's action on another person is called an externality by economists. We now extend the model of a jungle to allow for externalities. The following definition of a society differs from our [previous definition](#) in two respects. First, the preferences of each individual are defined over the set of [allocations](#) rather than the set of houses. Second, these preferences are not required to be strict. We allow non-strict preferences to include cases in which some individuals care only about the houses occupied by a specific group of individuals, possibly only herself. An individual who has such preferences is indifferent between allocations that differ only in the houses assigned to individuals outside the group.

Definition 8.6: Society with externalities

A *society with externalities* $\langle N, H, (\succsim^i)_{i \in N} \rangle$ consists of

individuals

a finite set N

houses

a finite set H with the same number of members as N

preferences

for each individual $i \in N$, a [preference relation](#) \succsim^i over the set of allocations.

The definition of a jungle with externalities differs from our [previous definition](#) only in that the society is replaced by a society with externalities.

Definition 8.7: Jungle with externalities

A *jungle with externalities* $\langle N, H, (\succsim^i)_{i \in N}, \triangleright \rangle$ consists of a [society with externalities](#) $\langle N, H, (\succsim^i)_{i \in N} \rangle$ and a power relation \triangleright , a complete, transitive, antisymmetric binary relation on the set N of individuals.

Despite the similarity of this definition with our original definition of a **jungle**, the meanings of the power relation \triangleright in the definitions differ. In a **jungle**, $i \triangleright j$ means simply that i can take over the house occupied by j . In a **jungle with externalities**, $i \triangleright j$ means that individual i can take over the house occupied by j and force j to occupy the house currently occupied by i . When considering whether to force such an exchange, individual i compares the current allocation with the one in which she and individual j exchange houses.

Definition 8.8: Equilibrium of jungle with externalities

An *equilibrium* of the **jungle with externalities** $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$ is an **allocation** a^* such that for no individuals $i, j \in N$ is it the case that $i \triangleright j$ and $b \succ^i a^*$, where b is the allocation that differs from a^* only in that $b(i) = a^*(j)$ and $b(j) = a^*(i)$.

The following examples show that the two basic results about the equilibrium of a jungle without externalities, its existence ([Proposition 8.1](#)) and its Pareto stability ([Proposition 8.4](#)), do not hold in a jungle with externalities.

Example 8.3: Nonexistence of equilibrium in jungle with externalities

Suppose $N = \{1, 2, 3\}$ and $H = \{\color{red}\blacktriangle, \color{blue}\blacktriangle, \color{green}\blacktriangle\}$, and think of the houses as being located on a circle, as in the following figure.



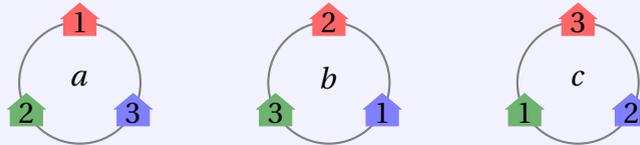
Assume that $1 \triangleright 2 \triangleright 3$ and suppose that individual 1 most prefers any of the three allocations in which she is individual 2's clockwise neighbor, and individual 2 most prefers any of the three allocations in which she is individual 1's clockwise neighbor.

In an equilibrium, 1 must be the clockwise neighbor of 2, because if 3 is the clockwise neighbor of 2, individual 1 can become the clockwise neighbor of 2 by forcing 3, who is less powerful than her, to exchange houses with her. Similarly, in an equilibrium, 2 must also be the clockwise neighbor of 1. But in no allocation is 1 the clockwise neighbor of 2 and 2 the clockwise neighbor of 1. Thus no equilibrium exists.

Example 8.4: Equilibria not Pareto stable in jungle with externalities

Consider a jungle that differs from the one in the previous example only in the individuals' preferences. Individual 1 most prefers to be individual 2's

clockwise neighbor, individual 2 most prefers to be 3's clockwise neighbor, and individual 3 most prefers to be 1's clockwise neighbor. Each individual prefers every allocation in which her clockwise neighbor is the individual she most prefers in that position to any other allocation. In addition, among the three allocations a , b , and c in which each individual has her favorite clockwise neighbor, shown in the following figures, each individual prefers a to b to c .



These three allocations are all equilibria, but b and c are not Pareto stable: all three individuals prefer a to both b and c . (For another example of an equilibrium in a jungle with externalities that is not Pareto stable, see Problem 6.)

Problems

1. *Jungle with preferences with indifference.* In a variant of a jungle in which the individuals' preferences are not necessarily strict, show the following results.
 - a. An equilibrium exists, but may not be unique.
 - b. An equilibrium may not be Pareto stable.
2. *Comparative statics.* Two jungles have the same set of individuals, the same set of houses, and the same preference profile (that is, the same society), but different power relations. The power relations differ only in that two individuals exchange positions. Compare the equilibria of the two jungles. Which individuals are necessarily assigned the same houses in both jungles? How are the individuals who exchange positions affected? How are the other individuals affected?
3. *Manipulability.* To prevent unnecessary clashes, the individuals in the jungle decide that a computer will calculate the equilibrium and they will abide by its recommendation. The computer is informed of the power relation, and each individual reports to the computer her preference relation. Then the computer calculates the equilibrium (given the reported preference relations), and the individuals abide by the allocation. Show that no individual can do better than reporting her true preference relation, regardless of the other individuals' reports.

4. *The jungle with insufficient housing.* Extend the model of the jungle to the case in which the number of houses is smaller than the number of individuals. In this case, an allocation is a function from the set of individuals to the set $H \cup \{\text{homeless}\}$ with the property that no two individuals are assigned to the same house. Each individual has a preference ordering over $H \cup \{\text{homeless}\}$. Assume, as before, that this ordering is strict; assume also that every individual prefers to be allocated any house than to be homeless. Define an equilibrium for this extended model and show that it always exists.
5. *A dynamic process.* Consider the following dynamic process for the jungle $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$.

At stage 0, all houses are vacant and no individual assigned to a house.

For any $t \geq 0$, given the assignment of individuals to houses at stage t , the assignment at stage $t + 1$ is defined as follows. Every individual goes to the house that is best for her among the houses that, at the end of stage t , are either unassigned or are assigned to her or to individuals weaker than her. For each house h , if only one individual goes to h , then she is assigned to h at the end of stage $t + 1$. If more than one individual goes to h , the strongest among the individuals at h are assigned to h and the other individuals are not assigned to any house at the end of stage $t + 1$.

Show that this process converges (at the latest at stage $|N|$) to an equilibrium of $\langle N, H, (\succ^i)_{i \in N}, \triangleright \rangle$.

6. *The cream guard.* An organization consists of n workers and n jobs, with $n \geq 3$. A manager, who is not one of the workers, assigns the workers to the jobs. The workers differ in their influence on the manager. If i has a larger influence than j , then i can persuade the manager to exchange i 's and j 's jobs. The nickname of one worker is "the cat"; she is at the bottom of the influence ladder. The jobs in the organization have an agreed ranking in terms of prestige, h_1, h_2, \dots, h_n , and every individual wants a job that is as prestigious as possible.

However, there is a small complication. The most prestigious job h_1 is the "cream guard". Every worker benefits from the situation in which the cat, rather than anyone else, guards the cream. Everyone is willing to sacrifice one place in the prestige ranking (but not more) to have the cat guard the cream.

- a. Find the equilibrium of this jungle with externalities.
- b. Explain why the equilibrium you found is not Pareto stable.

7. *Pareto instability and pairwise exchange.* Construct a society with an allocation that is not Pareto stable but for which no pair of individuals want to exchange their houses.
8. *Unique Pareto stable allocation.* Characterize the societies that have a unique Pareto stable allocation.
9. *Change requires majority approval.* Consider a variant of Pareto stability in which an allocation a is dominated by another allocation b if and only if $b(i) \succ^i a(i)$ for a strict majority of individuals. Construct a society in which no allocation is stable in this sense.
10. *Preference for strong neighbors.* Suppose $N = \{1, 2, \dots, 2k\}$ where $1 \triangleright 2 \triangleright \dots \triangleright 2k$ and $H = \{1a, 1b, 2a, 2b, \dots, ka, kb\}$. Every individual can force an individual who is weaker than her to exchange apartments. Every pair of apartments ma and mb for $m = 1, \dots, k$ are adjacent. Every individual has a preference relation over the pairs (h, j) , where h is the apartment she occupies and j is the occupant of the adjacent apartment.

Characterize the equilibrium of the *jungle with externalities* that models this situation under the assumption that every individual prefers (h, j) to (h', j') if and only if j is stronger than j' . That is, she cares only about the strength of her neighbor, not about the apartment she occupies. Is an equilibrium of this jungle Pareto stable?

Notes

This chapter is based on Piccione and Rubinstein (2007) and Rubinstein (2012, Chapter 3).