Models in Microeconomic Theory

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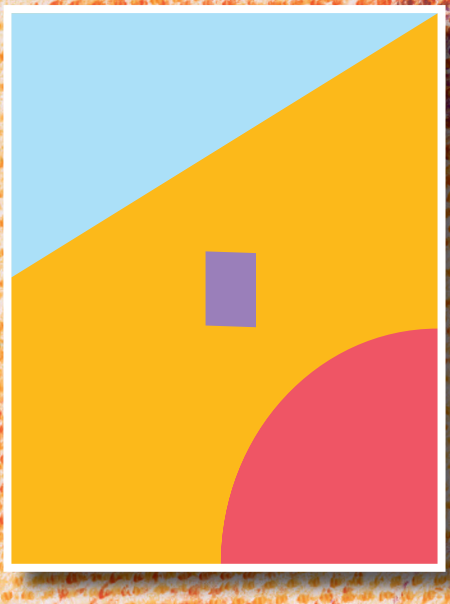

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3 Preferences under uncertainty

3.1 Lotteries

In Chapter 1 we discuss a model of preferences over an arbitrary set of alternatives. In this chapter we study an instance of the model in which an alternative in the set involves randomness regarding the consequence it yields. We refer to these alternatives as *lotteries*. For example, a raffle ticket that yields a car with probability 0.001 and nothing otherwise is a lottery. A vacation on which you will experience grey weather with probability 0.3 and sunshine with probability 0.7 can be thought of as a lottery as well.

The set X in the model we now discuss is constructed from a set Z of objects called prizes. A lottery specifies the probability with which each prize is realized. For simplicity, we study only lotteries for which the number of prizes that can be realized is finite.

Definition 3.1: Lotteries

Let Z be a set (of *prizes*). A *lottery over* Z is a function $p : Z \to \mathbb{R}$ that assigns a positive number (probability) $p(z)$ to a finite number of members of Z and 0 to all other members, with $\sum_{z \in Z} p(z) = 1$. The *support* of the lottery p, denoted $\text{supp}(p)$, is the set of all prizes to which p assigns positive probability, $\{z \in Z : p(z) > 0\}$.

We denote the set of all lotteries over Z by $L(Z)$, the lottery that yields the prize z with probability 1 by $[z]$, and the lottery that yields the prize z_k with probability α_k for $k = 1, \ldots, K$ by $\alpha_1 \cdot z_1 \oplus \alpha_2 \cdot z_2 \oplus \cdots \oplus \alpha_K \cdot z_K$.

If Z consists of two prizes, z_1 and z_2, then each member p of $L(Z)$ is specified by a pair (p_1, p_2) of nonnegative numbers with sum 1, where $p_1 = p(z_1)$ and $p_2 = p(z_2)$ are the probabilities of the prizes. Thus in this case $L(Z)$ can be identified with the blue line segment in Figure 3.1a. If Z includes three options, $L(Z)$ can similarly be identified with the triangle in Figure 3.1b.

3.2 Preferences over lotteries

We are interested in preference relations over $L(Z)$. In terms of the model in Chapter 1, the set X is equal to $L(Z)$. Here are some examples.

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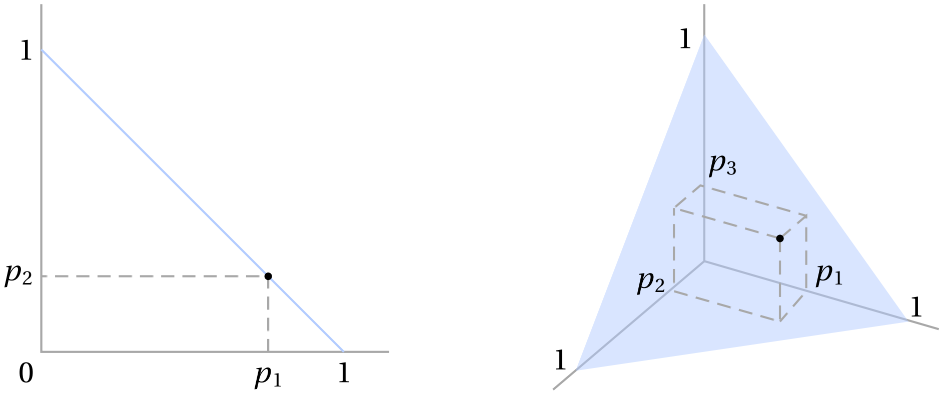

(a) The set $L(\{z_1, z_2\})$ of lotteries. The point (p_1, p_2) on the line segment represents the lottery p for which $p(z_1) = p_1$ and $p(z_2) = p_2$.

(b) The set $L(\{z_1, z_2, z_3\})$ of lotteries. The point (p_1, p_2, p_3) in the triangle represents the lottery p for which $p(z_1) = p_1$, $p(z_2) = p_2$, and $p(z_3) = p_3$.

Figure 3.1

Example 3.1: A pessimist

An individual has a strict preference relation $\succsim^*$ over the set Z of prizes and (pessimistically) evaluates lotteries by the worst prize, according to the preference relation, that occurs with positive probability. That is, she prefers the lottery $p \in L(Z)$ to the lottery $q \in L(Z)$ if she prefers the worst prize that occurs with positive probability in p to the worst prize that occurs with positive probability in q. Formally, define $w(p)$ to be a prize in $\text{supp}(p)$ such that $y \succsim^* w(p)$ for all $y \in \text{supp}(p)$. Then the individual's preference relation $\succsim$ over $L(Z)$ is defined by $p \succsim q$ if $w(p) \succsim^* w(q)$.

Note that there are many pessimistic preference relations, one for each preference relation over the set of prizes.

For any such preferences, the individual is indifferent between two lotteries whenever she is indifferent between the worst prizes that occur with positive probability in the lotteries. In one variant of the preferences that breaks this tie, if two lotteries share the same worst possible prize then the one for which the probability of the worst prize is lower is preferred.

Example 3.2: Good and bad

An individual divides the set Z of prizes into two subsets, *good* and *bad*. For any lottery $p \in L(Z)$, let $G(p) = \sum_{z \in good} p(z)$ be the total probability that a prize in *good* occurs. The individual prefers the lottery $p \in L(Z)$ to the lottery $q \in L(Z)$ if the probability of a prize in *good* occurring is at least as high for p as it is for q. Formally, $p \succsim q$ if $G(p) \geq G(q)$.

Different partitions of Z into *good* and *bad* generate different preference relations.

Example 3.3: Minimizing options

An individual wants the number of prizes that might be realized (the number of prizes in the support of the lottery) to be as small as possible. Formally, $p \succsim q$ if $|\text{supp}(p)| \leq |\text{supp}(q)|$. This preference relation makes sense for an individual who does not care about the realization of the lottery but wants to be as prepared as possible (physically or mentally) for all possible outcomes.

Preference relations over lotteries can take an unlimited number of other forms. To help us organize this large set, we now describe two plausible properties of preference relations and identify the set of all preference relations that satisfy the properties.

3.2.1 Properties of preferences

Continuity Suppose that for the prizes a, b, and c we have $[a] \succ [b] \succ [c]$, and consider lotteries of the form $\alpha \cdot a \oplus (1-\alpha) \cdot c$ (with $0 \leq \alpha \leq 1$). The continuity property requires that as we move continuously from $\alpha = 1$ (the degenerate lottery $[a]$, which is preferred to $[b]$) to $\alpha = 0$ (the degenerate lottery $[c]$, which is worse than $[b]$) we pass (at least once) some value of α such that the lottery $\alpha \cdot a \oplus (1-\alpha) \cdot c$ is indifferent to $[b]$.

Definition 3.2: Continuity

For any set Z of prizes, a preference relation $\succsim$ over $L(Z)$ is *continuous* if for any three prizes a, b, and c in Z such that $[a] \succ [b] \succ [c]$ there is a number α with $0 < \alpha < 1$ such that $[b] \sim \alpha \cdot a \oplus (1-\alpha) \cdot c$.

When Z includes at least three prizes, pessimistic preferences are not continuous: if $[a] \succ [b] \succ [c]$ then $[b] \succ \alpha \cdot a \oplus (1-\alpha) \cdot c$, for every number $\alpha < 1$. Good and bad preferences and minimizing options preferences satisfy the continuity condition vacuously because in each case there are no prizes a, b and c for which $[a] \succ [b] \succ [c]$.

Independence To define the second property, we need to first define the notion of a compound lottery. Suppose that uncertainty is realized in two stages. First the lottery p_k is drawn with probability α_k, for $k = 1,\ldots,K$, and then each prize z is realized with probability $p_k(z)$. In this case, the probability that each prize z is ultimately realized is $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$. Note that $\sum_{k=1,\ldots,K} \alpha_k p_k(z) \geq 0$ for each z and the sum of these expressions over all prizes z is equal to 1. We refer to the lottery in which each prize z occurs with probability $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$ as a compound lottery, and denote it by $\alpha_1 \cdot p_1 \oplus \cdots \oplus \alpha_K \cdot p_K$. For example, let $Z = \{W, D, L\}$, and define the lotteries $p = 0.6 \cdot W \oplus 0.4 \cdot L$ and $q = 0.2 \cdot W \oplus 0.3 \cdot D \oplus 0.5 \cdot L$. Then the compound lottery $\alpha \cdot p \oplus (1-\alpha) \cdot q$ is the lottery

$$(\alpha 0.6 + (1-\alpha)0.2) \cdot W \oplus ((1-\alpha)0.3) \cdot D \oplus (\alpha 0.4 + (1-\alpha)0.5) \cdot L.$$

Definition 3.3: Compound lottery

Let Z be a set of prizes, let $p_1,\ldots,p_K$ be lotteries in $L(Z)$, and let $\alpha_1,\ldots,\alpha_K$ be nonnegative numbers with sum 1. The *compound lottery* $\alpha_1 \cdot p_1 \oplus \cdots \oplus \alpha_K \cdot p_K$ is the lottery that yields each prize $z \in Z$ with probability $\sum_{k=1,\ldots,K} \alpha_k p_k(z)$.

We can now state the second property of preference relations over lotteries.

Definition 3.4: Independence

Let Z be a set of prizes. A preference relation $\succsim$ over $L(Z)$ satisfies the *independence property* if for any lotteries $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and $\beta \cdot a \oplus (1-\beta) \cdot b$ we have

$$[z_k] \succsim \beta \cdot a \oplus (1-\beta) \cdot b$$
$$\Leftrightarrow$$
$$\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$$
$$\succsim \alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot z_K.$$

The logic of the property is procedural: the only difference between the lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and the compound lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta a \oplus (1-\beta) b) \oplus \cdots \oplus \alpha_K \cdot z_K$ is in the kth term, which is z_k in the first case and $\beta \cdot a \oplus (1-\beta) \cdot b$ in the second case. Consequently it is natural to compare the two lotteries by comparing $[z_k]$ and $\beta \cdot a \oplus (1-\beta) \cdot b$.

Pessimistic preferences do not satisfy this property. Let $[a] \succ [b]$ and consider, for example, the lotteries

$$p = 0.6 \cdot a \oplus 0.4 \cdot b \quad \text{and} \quad q = 0.6 \cdot b \oplus 0.4 \cdot b = [b].$$

These lotteries differ only in the prize that is realized with probability 0.6. Given that $[a] \succ [b]$, the independence property requires that $p \succ q$. However, for a pessimist the two lotteries are indifferent since the worst prize in the lotteries is the same (b).

Minimizing options preferences also violate the independence property: for any prizes a and b, the lotteries $[a]$ and $[b]$ are indifferent, but $0.5 \cdot a \oplus 0.5 \cdot b \prec 0.5 \cdot b \oplus 0.5 \cdot b$.

Good and bad preferences satisfy the independence property. Let p be the lottery $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K$ and let q be the compound lottery

$$\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \cdots \oplus \alpha_K \cdot z_K.$$

Note that $G(p) - G(q) = \alpha_k G([z_k]) - \alpha_k G(\beta \cdot a \oplus (1-\beta) \cdot b)$, so that since $\alpha_k > 0$, the sign of $G(p) - G(q)$ is the same as the sign of $G([z_k]) - G(\beta \cdot a \oplus (1-\beta) \cdot b)$. Thus the preferences compare p and q in the same way that they compare $[z_k]$ and $\beta \cdot a \oplus (1-\beta) \cdot b$.

Monotonicity Consider lotteries that assign positive probability to only two prizes a and b, with $[a] \succ [b]$. We say that a preference relation over $L(Z)$ is monotonic if it ranks such lotteries by the probability that a occurs. That is, monotonic preferences rank lotteries of the type $\alpha \cdot a \oplus (1-\alpha) \cdot b$ according to the value of α.

The next result says that any preference relation over $L(Z)$ that satisfies the independence property is monotonic.

Lemma 3.1: Independence implies monotonicity

Let Z be a set of prizes. Assume that $\succsim$, a preference relation over $L(Z)$, satisfies the independence property. Let a and b be two prizes with $[a] \succ [b]$, and let α and β be two probabilities. Then

$$\alpha > \beta \quad \Leftrightarrow \quad \alpha \cdot a \oplus (1-\alpha) \cdot b \succ \beta \cdot a \oplus (1-\beta) \cdot b.$$

Proof

Let $p_\alpha = \alpha \cdot a \oplus (1-\alpha) \cdot b$. Because $\succsim$ satisfies the independence property, $p_\alpha \succ \alpha \cdot b \oplus (1-\alpha) \cdot b = [b]$. Using the independence property again we get

$$p_\alpha = (\beta/\alpha) \cdot p_\alpha \oplus (1-\beta/\alpha) \cdot p_\alpha \succ (\beta/\alpha) \cdot p_\alpha \oplus (1-\beta/\alpha) \cdot b = \beta \cdot a \oplus (1-\beta) \cdot b.$$

3.3 Expected utility

We now introduce the type of preferences most commonly assumed in economic theory. These preferences emerge when an individual uses the following scheme to compare lotteries. She attaches to each prize z a number, which we refer to as the value of the prize (or the Bernoulli number) and denote $v(z)$; when evaluating a lottery p, she calculates the expected value of the lottery, $\sum_{z\in Z} p(z)v(z)$. The individual's preferences are then defined by

$$p \succsim q \quad \text{if} \quad \sum_{z\in Z} p(z)v(z) \geq \sum_{z\in Z} q(z)v(z).$$

Definition 3.5: Expected utility

For any set Z of prizes, a preference relation $\succsim$ on the set $L(Z)$ of lotteries is *consistent with expected utility* if there is a function $v: Z \to \mathbb{R}$ such that $\succsim$ is represented by the utility function U defined by $U(p) = \sum_{z\in Z} p(z)v(z)$ for each $p \in L(Z)$. The function v is called the *Bernoulli function* for the representation.

We first show that a preference relation consistent with expected utility is continuous and satisfies the independence property.

Proposition 3.1: Expected utility is continuous and independent

A preference relation on a set of lotteries that is consistent with expected utility satisfies the continuity and independence properties.

Proof

Let Z be a set of prizes, let $\succsim$ be a preference relation over $L(Z)$, and let $v: Z \to \mathbb{R}$ be a function such that the function U defined by $U(p) = \sum_{z\in Z} p(z)v(z)$ for each $p \in L(Z)$ represents $\succsim$.

Continuity Let a, b, and $c \in Z$ satisfy $[a] \succ [b] \succ [c]$. For every $z \in Z$, $U([z]) = v(z)$. Thus $v(a) > v(b) > v(c)$. Let α satisfy $\alpha v(a) + (1-\alpha)v(c) = v(b)$ (that is, $0 < \alpha = (v(b) - v(c))/(v(a) - v(c)) < 1$). Then $\alpha \cdot a \oplus (1-\alpha)\cdot c \sim [b]$.

Independence Consider lotteries $\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_K \cdot z_K$ and $\beta \cdot a \oplus (1-\beta)\cdot b$. We have

$$\begin{aligned} &\alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot z_k \oplus \cdots \oplus \alpha_K \cdot z_K \\ &\qquad \succsim \alpha_1 \cdot z_1 \oplus \cdots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta)\cdot b) \oplus \cdots \oplus \alpha_K \cdot q_K \\ &\Leftrightarrow \text{(by the formula for } U\text{, which represents } \succsim \text{)} \end{aligned}$$

$$
\begin{aligned}
&\alpha_1 v(z_1)+\cdots+\alpha_k v(z_k)+\cdots+\alpha_K v(z_K)\\
&\qquad\geq \alpha_1 v(z_1)+\cdots+\alpha_k\beta v(a)+\alpha_k(1-\beta)v(b)+\cdots+\alpha_K v(z_K)\\
&\qquad\qquad\Longleftrightarrow \text{(by algebra)}\\
&\qquad\qquad\alpha_k v(z_k)\geq \alpha_k\beta v(a)+\alpha_k(1-\beta)v(b)\\
&\qquad\qquad\Longleftrightarrow \text{(since } \alpha_k>0)\\
&\qquad\qquad v(z_k)\geq \beta v(a)+(1-\beta)v(b)\\
&\qquad\qquad\Longleftrightarrow \text{(by the formula for } U\text{, which represents } \succsim)\\
&\qquad\qquad [z_k]\succsim \beta\cdot a\oplus(1-\beta)\cdot b.
\end{aligned}
$$

The next result, the main one of this chapter, shows that any preference relation that satisfies continuity and independence is consistent with expected utility. That is, we can attach values to the prizes such that the comparison of the expected values of any two lotteries is equivalent to the comparison of the lotteries according to the preference relation.

Proposition 3.2: Continuity and independence implies expected utility

A preference relation on a set of lotteries with a finite set of prizes that satisfies the continuity and independence properties is consistent with expected utility.

Proof

Let Z be a finite set of prizes and let $\succsim$ be a preference relation on $L(Z)$ satisfying continuity and independence. Label the members of Z so that $[z_1]\succsim\cdots\succsim[z_K]$. Let $z_1=M$ (the best prize) and $z_K=m$ (the worst prize).

First suppose that $[M]\succ[m]$. Then by continuity, for every prize z there is a number $v(z)$ such that $[z]\sim v(z)\cdot M\oplus(1-v(z))\cdot m$. In fact, by monotonicity this number is unique. Consider a lottery $p(z_1)\cdot z_1\oplus\cdots\oplus p(z_K)\cdot z_K$. By applying independence K times, the individual is indifferent between this lottery and the compound lottery

$$p(z_1)\cdot\big(v(z_1)\cdot M\oplus(1-v(z_1))\cdot m\big)\oplus\cdots\oplus p(z_K)\cdot\big(v(z_K)\cdot M\oplus(1-v(z_K))\cdot m\big).$$

This compound lottery is equal to the lottery

$$\Big(\sum_{k=1,\dots,K} p(z_k)v(z_k)\Big)\cdot M\oplus\Big(1-\sum_{k=1,\dots,K} p(z_k)v(z_k)\Big)\cdot m.$$

Given $[M]\succ[m]$, Lemma 3.1 implies that the comparison between the lotteries p and q is equivalent to the comparison between the numbers $\sum_{k=1,\dots,K} p(z_k)v(z_k)$ and $\sum_{k=1,\dots,K} q(z_k)v(z_k)$.

Now suppose that $[M] \sim [m]$. Then by independence, $p \sim [M]$ for any lottery p. That is, the individual is indifferent between all lotteries. In this case, choose $v(z_k) = 0$ for all k. Then the function U defined by $U(p) = \sum_{z \in Z} p(z)v(z) = 0$ for each $p \in L(Z)$ represents the preference relation.

Comment

Note that if the function $v : Z \to \mathbb{R}$ is the Bernoulli function for an expected utility representation of a certain preference relation over $L(Z)$ then for any numbers $\alpha > 0$ and β so too is the function w given by $w(z) = \alpha v(z) + \beta$ for all $z \in Z$. In fact the converse is true also (we omit a proof): if $v : Z \to \mathbb{R}$ and $w : Z \to \mathbb{R}$ are Bernoulli functions for representations of a certain preference relation then for some numbers $\alpha > 0$ and β we have $w(z) = \alpha v(z) + \beta$ for all $z \in Z$.

3.4 Theory and experiments

We now briefly discuss the connection (and disconnection) between the model of expected utility and human behavior. The following well-known pair of questions demonstrates a tension between the two.

> Imagine that you have to choose between the following two lotteries.
>
> L_1: you receive \$4,000 with probability 0.2 and zero otherwise.
> R_1: you receive \$3,000 with probability 0.25 and zero otherwise.
>
> Which lottery do you choose?

> Imagine that you have to choose between the following two lotteries.
>
> L_2: you receive \$4,000 with probability 0.8 and zero otherwise.
> R_2: you receive \$3,000 with certainty.
>
> Which lottery do you choose?

The responses to these questions by 7,932 students at http://gametheory.tau.ac.il are summarized in the following table.

	L_2	R_2
L_1	20%	44%
R_1	5%	31%

In our notation, the lotteries are

$$L_1 = 0.2\cdot[\$4000]\oplus 0.8\cdot[\$0] \quad\text{and}\quad R_1 = 0.25\cdot[\$3000]\oplus 0.75\cdot[\$0]$$
$$L_2 = 0.8\cdot[\$4000]\oplus 0.2\cdot[\$0] \quad\text{and}\quad R_2 = [\$3000].$$

Note that $L_1 = 0.25\cdot L_2 \oplus 0.75\cdot[0]$ and $R_1 = 0.25\cdot R_2 \oplus 0.75\cdot[0]$. Thus if a preference relation on $L(Z)$ satisfies the independence property, it should rank L_1 relative to R_1 in the same way that it ranks L_2 relative to R_2. So among individuals who have a strict preference between the lotteries, only those whose answers are (*i*) L_1 and L_2 or (*ii*) R_1 and R_2 have preferences that can be represented by expected utility. About 51% of the participants are in this category.

Of the rest, very few (5%) choose R_1 and L_2. The most popular pair of answers is L_1 and R_2, chosen by 44% of the participants. Nothing is wrong with those subjects (which include the authors of this book). But such a pair of choices conflicts with expected utility theory; the conflict is known as the Allais paradox.

One explanation for choosing R_2 over L_2 is that the chance of getting an extra \$1,000 is not worth the risk of losing the certainty of getting \$3,000. The idea involves risk aversion, which we discuss in the next section.

Many of us use a different consideration when we compare L_1 and R_1. There, we face a dilemma: increasing the probability of winning versus a significant loss in the prize. The probabilities 0.25 and 0.2 seem similar whereas the prizes \$4,000 and \$3,000 are not. Therefore, we ignore the difference in the probabilities and focus on the difference in the prizes, a consideration that pushes us to choose L_1.

Experimentalists usually present the two questions to different groups of people, randomly assigning each participant to one of the questions. They do so to avoid participants guessing the object of the experiment, in which case a participant's answer to the second question might be affected by her answer to the first one. However, even when the two questions are given to the same people, we get similar results.

Findings like the ones we have described have led to many suggestions for alternative forms of preferences over the set of lotteries. In experiments, the behavior of many people is inconsistent with any of these alternatives; each theory seems at best to fit some people's behavior in some contexts.

3.5 Risk aversion

We close the chapter by considering attitudes to risk. We assume that the set Z of prizes is the set of nonnegative real numbers, and think of the prize z as

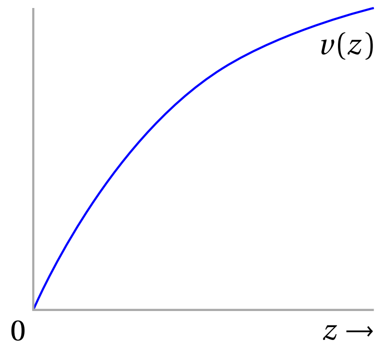

Figure 3.2 A concave Bernoulli function.

the monetary reward of $\$z$. We denote the expected value of any lottery p by $E(p) = \sum_{z \in \text{supp}(Z)} p(z)z$.

An individual is risk-neutral if she cares only about the expectation of a lottery, so that her preferences over lotteries are represented by $E(p)$. Such preferences are consistent with expected utility—take $v(z) = z$. An individual is risk-averse if for every lottery p she finds the prize equal to the expectation of p at least as good as p. That is, an individual with preference relation $\succsim$ is risk-averse if $[E(p)] \succsim p$ for every p. If for every lottery p that involves more than one prize, the individual strictly prefers $[E(p)]$ to p, she is strictly risk-averse.

Definition 3.6: Risk aversion and risk neutrality

If $Z = \mathbb{R}_+$, a preference relation $\succsim$ on the set $L(Z)$ of lotteries over Z is *risk-averse* if $[E(p)] \succsim p$ for every lottery $p \in L(Z)$, is *strictly risk-averse* if $[E(p)] \succ p$ for every lottery $p \in L(Z)$ that involves more than one prize, and is *risk-neutral* if $[E(p)] \sim p$ for every lottery $p \in L(Z)$, where $E(p) = \sum_{z \in Z} p(z)z$.

A strictly risk-averse individual is willing to pay a positive amount of money to replace a lottery with its expected value, so that the fact that an individual buys insurance (which typically reduces but does not eliminate risk) suggests that her preferences are strictly risk-averse. On the other hand, the fact that an individual gambles, paying money to replace a certain amount of money with a lottery with a lower expected value, suggests that her preferences are not risk-averse.

The property of risk aversion applies to any preference relation, whether or not it is consistent with expected utility. We now show that if an individual's preference relation is consistent with expected utility, it is risk-averse if and only if it has a representation for which the Bernoulli function is concave. (Refer to Figure 3.2.)

Proposition 3.3: Risk aversion and concavity of Bernoulli function

Let $Z = \mathbb{R}_+$, assume $\succsim$ is a preference relation over $L(Z)$ that is consistent with expected utility, and let v be the Bernoulli function for the representation. Then $\succsim$ is risk-averse if and only if v is concave.

Proof

Let x and y be any prizes and let $\alpha \in [0,1]$. If $\succsim$ is risk-averse then $[\alpha x + (1-\alpha)y] \succsim \alpha \cdot x \oplus (1-\alpha) \cdot y$, so that $v(\alpha x + (1-\alpha)y) \geq \alpha v(x) + (1-\alpha)v(y)$. That is, v is concave.

Now assume that v is concave. Then Jensen's inequality implies that $v\left(\sum_{z \in Z} p(z)z\right) \geq \sum_{z \in Z} p(z)v(z)$, so that $\left[\sum_{z \in Z} p(z)z\right] \succsim p$. Thus the individual is risk-averse.

Problems

1. *Most likely prize.* An individual evaluates a lottery by the probability that the most likely prize is realized (independently of the identity of the prize). That is, for any lotteries p and q we have $p \succsim q$ if $\max_z p(z) \geq \max_z q(z)$. Such a preference relation is reasonable in a situation where the individual is indifferent between all prizes (e.g., the prizes are similar vacation destinations) and she can prepare herself for only one of the options (in contrast to Example 3.3, where she wants to prepare herself for all options and prefers a lottery with a smaller support).

   Show that if Z contains at least three elements, this preference relation is continuous but does not satisfy independence.

2. *A parent.* A parent has two children, A and B. The parent has in hand only one gift. She is indifferent between giving the gift to either child but prefers to toss a fair coin to determine which child obtains the gift over giving it to either of the children.

   Explain why the parent's preferences are not consistent with expected utility.

3. *Comparing the most likely prize.* An individual has in mind a preference relation $\succsim^*$ over the set of prizes. Whenever each of two lotteries has a single most likely prize she compares the lotteries by comparing the most likely prizes using $\succsim^*$. Assume Z contains at least three prizes. Does such a preference relation satisfy continuity or independence?

4. *Two prizes.* Assume that the set Z consists of two prizes, a and b. Show that only three preference relations over $L(Z)$ satisfy independence.

5. *Simple lotteries.* Let Y be a finite set of objects. For any number $\alpha \in [0,1]$ and object $z \in Y$, the *simple lottery* (α, z) means that z is obtained with probability α and nothing is obtained with probability $1-\alpha$. Consider preference relations over the set of simple lotteries.

   A preference relation satisfies A1 if for every x, $y \in Y$ with $(1,y) \succ (1,x)$ there is a probability α such that $(\alpha, y) \sim (1,x)$.

   A preference relation satisfies A2 if when $\alpha \geq \beta$ then for any x, $y \in Y$ the comparison between (α, x) and (β, y) is the same as that between $(1,x)$ and $(\beta/\alpha, y)$.

   a. Show that if an individual has in mind a function v that attaches a number $v(z) > 0$ to each object z and her preference relation $\succsim$ is defined by $(\alpha, x) \succsim (\beta, y)$ if $\alpha v(x) \geq \beta v(y)$, then the preference relation satisfies both A1 and A2.

   b. Suggest a preference relation that satisfies A1 but not A2 and one that satisfies A2 but not A1.

The following questions refer to the model of expected utility with monetary prizes and risk aversion described in Section 3.5. For these questions, consider a risk-averse individual whose preferences are consistent with expected utility. A prize is the total amount of money she holds after she makes a choice and after the realization of the uncertainties. Denote by v a Bernoulli function whose expected value represents the individual's preferences over $L(Z)$ and assume that v has a derivative.

7. *Additional lottery.* An individual faces the monetary lottery p. She is made the following offer. For each realization of the lottery another lottery will be executed according to which she will win an additional dollar with probability $\frac{1}{2}$ and lose a dollar with probability $\frac{1}{2}$. Describe the lottery q that she faces if she accepts the offer and show that if she is strictly risk-averse she rejects the offer.

8. *Casino.* An individual has wealth w and has to choose an amount x, after which a lottery is conducted in which with probability α she gets $2x$ and with probability $1-\alpha$ she loses x. Show that the higher is α the higher is the amount x she chooses.

9. *Insurance.* An individual has wealth w and is afraid that an accident will occur with probability p that will cause her a loss of D. The individual has to choose an amount, x, she will pay for insurance that will pay her λx (for some given λ) if the accident occurs.

   a. The insurer's expected profit is $x - \lambda p x$. Assume that λ makes this profit zero, so that $\lambda = 1/p$. Show that if the individual is risk-averse she optimally chooses $x = pD$, so that she is fully insured: her net wealth is the same whether or not she has an accident.

   b. Assume that $p\lambda < 1$ (that is, the insurer's expected profit is positive). Show that if the individual is strictly risk-averse then she chooses partial insurance: $\lambda x < D$.

Notes

The theory of expected utility was developed by von Neumann and Morgenstern (1947, 15–29 and 617–628). The Allais paradox (Section 3.4) is due to Allais (1953, 527). The notion of risk aversion (Section 3.5) is due to Pratt (1964). The exposition of the chapter draws upon Rubinstein (2006a, Lecture 7).