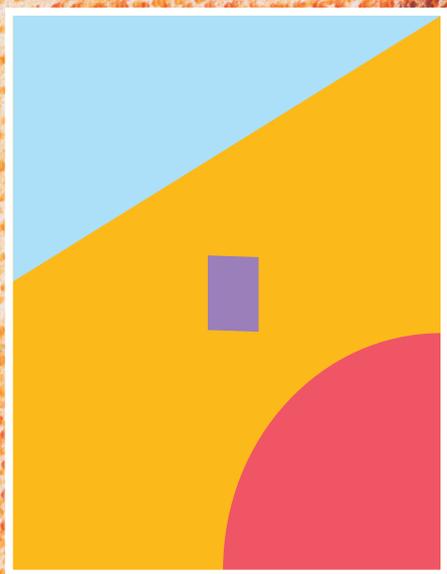


Models in Microeconomic Theory

Martin J. Osborne
Ariel Rubinstein





<https://www.openbookpublishers.com>

© 2020 Martin J. Osborne and Ariel Rubinstein



This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivs license (CC BY-NC-ND 4.0). This license allows you to share, copy, distribute, and transmit the work providing you do not modify the work, you do not use the work for commercial purposes, you attribute the work to the authors, and you provide a link to the license. Attribution should not in any way suggest that the authors endorse you or your use of the work and should include the following information.

Martin J. Osborne and Ariel Rubinstein, *Models in Microeconomic Theory*. Cambridge, UK: Open Book Publishers, 2020, <https://doi.org/10.11647/OBP.0204>.

To access detailed and updated information on the license, please visit <https://www.openbookpublishers.com/product/1159#copyright>.

Further details about CC BY licenses are available at <https://creativecommons.org/licenses/by-nc-nd/4.0/>. All external links were active at the time of publication unless otherwise stated and have been archived via the Internet Archive Wayback Machine at <https://archive.org/web>.

Updated digital material and resources associated with this book are available at <https://www.openbookpublishers.com/product/1159#resources>.

Every effort has been made to identify and contact copyright holders and any omission or error will be corrected if notification is made to the publisher.

ISBN Paperback: 978-1-78374-920-1

ISBN Hardback: 978-1-78374-921-8

ISBN Digital (PDF): 978-1-78374-922-5

DOI: 10.11647/OBP0204

This book is available in two versions, one that uses feminine pronouns and one that uses masculine pronouns. This version uses masculine pronouns.

Version: 2020.12.21 (h)

11 Variants of an exchange economy

In this chapter we study two variants of the model of an exchange economy that demonstrate the richness of the model. In the first variant we insert into the framework the basic model of supply and demand for a single indivisible good (with which you may be familiar from an introductory course in economics). In the second variant we use the framework to capture a situation in which individuals face uncertainty about the future resources. This variant is used to analyze markets for insurance and bets.

11.1 Market with indivisible good and money

A single indivisible good is traded in a market for money. Each person can consume either one unit of the good, or none of it. Consuming more than one unit, or a fraction of a unit, is impossible. A ticket for a performance and membership in a club are examples of such goods. Some people initially own one unit of the good and some do not. Every person is characterized by the monetary value he assigns to having one unit of the good. There is room for trade if the value assigned by some person who initially has the good is lower than the value assigned by some person who does not initially have the good. In that case, many transactions may be mutually beneficial for the pair of people. We are interested in who buys the good, who sells it, and the prices at which the transactions take place.

11.1.1 Model

The model is a variant of an [exchange economy](#) with two goods. Good 1 is money, which can be held in any nonnegative amount, and good 2 is an indivisible good, which can be held (1) or not held (0). Thus a bundle is a pair (x_1, x_2) , where x_1 is a nonnegative number and x_2 is either 0 or 1. Formally, the set of possible bundles of goods (which is \mathbb{R}_+^2 in the previous chapter) is

$$X = \{(x_1, x_2) : x_1 \in \mathbb{R}_+ \text{ and } x_2 \in \{0, 1\}\} = \mathbb{R}_+ \times \{0, 1\}.$$

We assume that the preferences over X of each individual i are represented by the function $x_1 + v^i x_2$ where $v^i \geq 0$. Thus individual i prefers the bundle $(s, 1)$ to

the bundle $(t, 0)$ if and only if $s + v^i > t$. That is, he prefers holding the indivisible good to not owning it if and only if he has to give up less than v^i units of money to obtain it. We refer to v^i as i 's *valuation* of the good.

We assume that every individual who does not own the indivisible good initially has enough money to pay his valuation to obtain the good: no individual is cash constrained. That is, for every individual i whose initial bundle $e(i)$ has $e_2(i) = 0$ we assume that $e_1(i) \geq v^i$.

Definition 11.1: Exchange economy with indivisible good and money

An *exchange economy with an indivisible good and money* $\langle N, (v^i)_{i \in N}, e \rangle$ has the following components.

Individuals

A finite set N .

Valuations

For each individual $i \in N$, a nonnegative number v^i (i 's *valuation* of the good); the preference relation of each individual i over $X = \mathbb{R}_+ \times \{0, 1\}$ is represented by the function u^i defined by $u^i(x_1, x_2) = x_1 + v^i x_2$.

Initial allocation

A function e that assigns to each individual i a bundle $e(i) \in X$, the bundle that i initially owns, with $e_1(i) \geq v^i$ if $e_2(i) = 0$.

Individual i is a (potential) *buyer* if $e_2(i) = 0$ and a (potential) *seller* if $e_2(i) = 1$. To avoid degenerate cases, we assume that the economy contains at least one buyer and one seller.

As for an exchange economy studied in the previous chapter, we assume that a single price for the indivisible good prevails. No individual has the power to influence the price and every individual believes that he can trade the good at this price, and only at this price.

The assumption that all transactions take place at the same price is not obviously reasonable. Consider, for example, the economy that consists of two buyers, B_4 and B_{10} , with valuations 4 and 10, and two sellers, S_0 and S_6 , with valuations 0 and 6. If first B_{10} meets S_6 they may trade at a price between 6 and 10. If, subsequently, B_4 meets S_0 they may trade at a price between 0 and 4. Whether such a sequence of transactions occurs might depend on the information available to the individuals about other individuals' valuations and the pattern in which they meet. For example, if B_{10} realizes that S_0 is about to sell the good at a price of at most 4, he might approach S_0 and offer him a price between 4 and 6. The concept of competitive equilibrium that we study in this chapter

does not model the formation of prices; it simply assumes that somehow a price emerges and becomes known to all individuals.

As the equilibrium notion we adapt the concept of **competitive equilibrium** for an exchange economy. We set the price of money to be 1. Thus a price system is a pair $(1, p)$, where p is the amount of money transferred from a buyer to a seller in exchange for the indivisible good. The budget set of each seller i contains two bundles, $(e_1(i), 1)$ (he retains the good) and $(e_1(i) + p, 0)$ (he sells the good). He optimally sells the good if $p > v^i$, and is indifferent between selling and not if $p = v^i$. Similarly, the budget set of each buyer i contains two bundles, $(e_1(i) - p, 1)$ (he buys the good) and $(e_1(i), 0)$ (he does not). He optimally buys the good if $p < v^i$, and is indifferent between buying and not if $p = v^i$. A price p is an equilibrium price if the number of units buyers wish to purchase is equal to the number of units sellers wish to sell.

Definition 11.2: Competitive equilibrium of exchange economy with indivisible good and money

A *competitive equilibrium* of the **exchange economy with an indivisible good and money** $\langle N, (v^i)_{i \in N}, e \rangle$ is a pair (p, a) where

- p is a nonnegative number (the price of the indivisible good)
- $a = (a(i))_{i \in N}$ is a profile of bundles

such that

optimality of choices

for each individual i , (i) $a_2(i) = 1$ if $p < v^i$, (ii) $a_2(i) = 0$ if $p > v^i$, and (iii) $a_1(i) = e_1(i) + p(e_2(i) - a_2(i))$.

feasibility

$$\sum_{i \in N} a(i) = \sum_{i \in N} e(i).$$

Notice the analogue of **Proposition 10.1** for this model: if the total amount of the indivisible good demanded by all individuals is equal to the total amount available, that is, $\sum_{i \in N} a_2(i) = \sum_{i \in N} e_2(i)$, then the total amount of money demanded by all individuals, $\sum_{i \in N} (e_1(i) + p(e_2(i) - a_2(i)))$, is equal to $\sum_{i \in N} e_1(i)$, the total amount of money available.

Consider the economy with four individuals we specified earlier. Every price p with $4 \leq p \leq 6$ is part of an equilibrium, in which B_{10} buys the good, S_0 sells the good, and the other two individuals refrain from trade. A price greater than 6 is not part of an equilibrium since for such a price at most one individual, B_{10} , wants to have the indivisible good but two units of it are available. By a similar argument, a price less than 4 is not part of an equilibrium.

Example 11.1

An exchange economy with an indivisible good and money contains 14 sellers with valuation 0 and 17 buyers with valuation 100.

This economy has no equilibrium with a price less than 100, because at such a price all 17 buyers optimally choose to have the good, but only 14 units of the good are available. Also the economy has no equilibrium with a price greater than 100, because at such a price no individual wants to have the good. Thus the only possible equilibrium price is 100. At this price, all 14 sellers optimally wish to sell the good and every buyer is indifferent between buying and not buying the good. Therefore the price 100 together with a profile of choices in which every seller sells his unit, 14 of the 17 buyers choose to buy a unit, and the remaining 3 buyers chooses not to do so, is a competitive equilibrium.

Note that a competitive equilibrium may involve no trade. If the valuation of every seller exceeds the valuation of every buyer, in no competitive equilibrium does any trade take place; an equilibrium price is any number between the lowest valuation among the sellers and the highest valuation among the buyers.

The following result proves that competitive equilibrium exists and characterizes all equilibria.

Proposition 11.1: Characterization of competitive equilibrium

Let $\langle N, (v^i)_{i \in N}, e \rangle$ be an **exchange economy with an indivisible good and money**. Denote the number of individuals by n and name them so that $v^1 \geq v^2 \geq \dots \geq v^n$. Denote by s the number of sellers (equal to the number of units of the indivisible good available). A number p is a **competitive equilibrium price** for the economy if and only if $v^{s+1} \leq p \leq v^s$.

Proof

Let $p \in [v^{s+1}, v^s]$ and define the allocation a as follows.

individuals	$a(i)$
buyers $i \in \{1, \dots, s\}$	$(e_1(i) - p, 1)$
sellers $i \in \{1, \dots, s\}$	$(e_1(i), 1)$
buyers $i \in \{s+1, \dots, n\}$	$(e_1(i), 0)$
sellers $i \in \{s+1, \dots, n\}$	$(e_1(i) + p, 0)$

The optimality condition is satisfied since any individual whose valuation is greater than p is in $\{1, \dots, s\}$ and any individual whose valuation is less

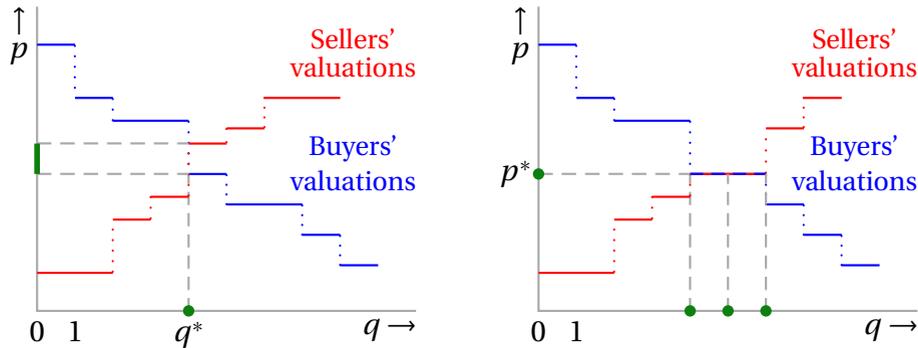


Figure 11.1 Equilibria of exchange economies with an indivisible good and money. The red lines represent the sellers' valuations and the blue lines represent the buyers' valuations. The green line segment and disk on the vertical axis represent the equilibrium prices and the green disks on the horizontal axis represent the number of units traded in an equilibrium.

than p is in $\{s+1, \dots, n\}$. The allocation is feasible because $\sum_{i \in N} a_2(i) = s = \sum_{i \in N} e_2(i)$. Thus (p, a) is a competitive equilibrium.

A price greater than v^s is not an equilibrium price because for such a price the number of individuals who optimally hold the good is less than s . Similarly, a price less than v^{s+1} is not an equilibrium price.

Comments

1. This result is illustrated in [Figure 11.1](#). The blue lines show the buyers' valuations, plotted in descending order. The length of each solid line segment is the number of buyers whose valuations are equal to the height of the segment. The red lines similarly show the sellers' valuations, plotted in ascending order. In the left panel economy, there is a range of competitive equilibrium prices, indicated in green; in every equilibrium the total amount of the good traded is q^* . In the right panel economy, there is a unique competitive equilibrium price p^* and a range of possible equilibrium quantities.
2. The result implies that an economy has a unique equilibrium price if and only if $v^s = v^{s+1}$. In this case, as in the right-hand panel of [Figure 11.1](#), the number of equilibrium transactions is not unique. For example, if s is a seller and $s+1$ is a buyer, then there is an equilibrium in which these two trade, and also an equilibrium in which they do not.

We now characterize the Pareto stable allocations and prove that, as in the model of the previous chapter, every competitive equilibrium allocation is [Pareto stable](#).

Proposition 11.2: Pareto stable allocations

Let $\langle N, (v^i)_{i \in N}, e \rangle$ be an exchange economy with an indivisible good and money. (a) An allocation a is Pareto stable if and only if $v^i \geq \min\{v^j, a_1(j)\}$ for any pair (i, j) of individuals in which i holds the good ($a_2(i) = 1$) and j does not ($a_2(j) = 0$). (b) Every competitive equilibrium allocation is Pareto stable.

Proof

(a) Consider an allocation a which in which $a_2(i) = 1$, $a_2(j) = 0$, and $v^i < \min\{v^j, a_1(j)\}$. Let b be the allocation identical to a except that $b(i) = (a_1(i) + \delta, 0)$ and $b(j) = (a_1(j) - \delta, 1)$ where $v^i < \delta < \min\{v^j, a_1(j)\}$. Then b is feasible and Pareto dominates a , so that a is not Pareto stable.

Now let a be an allocation such that for any pair of individuals i and j for which $a_2(i) = 1$ and $a_2(j) = 0$, we have $v^i \geq \min\{v^j, a_1(j)\}$. We argue that a is Pareto stable.

Suppose the allocation b Pareto dominates a . By the feasibility of a and b , the number k of individuals who hold the good in a but not in b is equal to the number of individuals who hold the good in b but not in a (refer to Figure 11.2). Denote by v the lowest valuation of an individual who holds the good in a .

If $b_2(i) = a_2(i)$ then $b_1(i) \geq a_1(i)$. For each of the k individuals for whom $a_2(i) = 1$ and $b_2(i) = 0$, we have $b_1(i) \geq a_1(i) + v^i \geq a_1(i) + v$. For each of the k individuals for whom $a_2(i) = 0$ and $b_2(i) = 1$ we have $b_1(i) + v^i \geq a_1(i)$ and $b_1(i) \geq 0$. Thus $b_1(i) \geq a_1(i) - \min\{v^i, a_1(i)\}$. By the assumption on a , $v^j \geq \min\{v^i, a_1(i)\}$ for all j who hold the good in a so $v \geq \min\{v^i, a_1(i)\}$ and thus $b_1(i) \geq a_1(i) - v$. For at least one individual the inequality is strict, so that $\sum_{i \in N} b_1(i) > \sum_{i \in N} a_1(i)$, contradicting the feasibility of b . Thus no allocation Pareto dominates a .

(b) Let (p^*, a) be a competitive equilibrium. Then for every individual i who holds the good $v^i \geq p^*$, and for every individual j who does not hold the good $v^j \leq p^*$. Then (a) implies that a is Pareto stable.

Recall that an allocation is in the core of an exchange economy if no subset of individuals can secede from the economy and allocate their initial bundles (in this case, units of the good and money) between themselves so that they are all better off. Proposition 10.5, showing that a competitive equilibrium allocation is in the core of an exchange economy, holds also for an exchange economy with an indivisible good and money (as you can verify). Further, for such an economy,

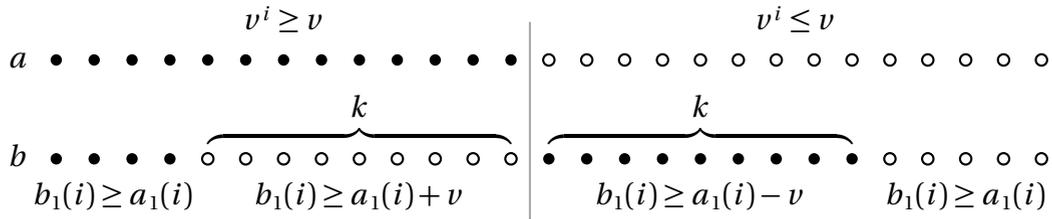


Figure 11.2 An illustration of the argument in the second part of the proof of part (a) of Proposition 11.2. Each disk represents an individual who holds the indivisible good, and each circle represents an individual who does not hold the good.

a stronger result is true: every core allocation is a competitive equilibrium allocation, so that the core is exactly the set of competitive allocations.

Proposition 11.3: Core and competitive equilibrium

For every allocation a in the **core** of an **exchange economy with an indivisible good and money** there is a number p such that (p, a) is a **competitive equilibrium** of the economy.

Proof

Denote the economy $\langle N, (v^i)_{i \in N}, e \rangle$. We have $a_1(i) + v^i a_2(i) \geq e_1(i) + v^i e_2(i)$ for every individual i since otherwise i can improve upon a by himself. Also, if $a_2(i) = e_2(i)$ then $a_1(i) = e_1(i)$ because if $a_1(i) > e_1(i)$ then the set $N \setminus \{i\}$ of individuals can improve upon a (it has the same amount of the indivisible good in a and e but has less money in a).

If $a = e$ then the valuation of every individual who holds the good in a is at least as high as the valuation of any individual who does not hold the good, since otherwise such a pair can improve upon a (given the assumption that each buyer i has at least v^i units of money). In this case (p, a) is an equilibrium for any p with $\max_{i \in N} \{v^i : a_2(i) = 0\} \leq p \leq \min_{i \in N} \{v^i : a_2(i) = 1\}$.

Now suppose $a \neq e$. Let $B = \{i \in N : e_2(i) = 0 \text{ and } a_2(i) = 1\}$ and $S = \{i \in N : e_2(i) = 1 \text{ and } a_2(i) = 0\}$. Since $a \neq e$ both B and S are nonempty and by the feasibility of a they have the same size. For every other individual i we have $a(i) = e(i)$.

If $a_1(j) - e_1(j) < e_1(i) - a_1(i)$ for some $j \in S$ and $i \in B$ (seller j receives less than buyer i pays) then for any number p with $a_1(j) - e_1(j) < p < e_1(i) - a_1(i)$ the set $\{i, j\}$ can improve upon a with the bundles $(e_1(i) - p, 1)$ for i and $(e_1(j) + p, 0)$ for j .

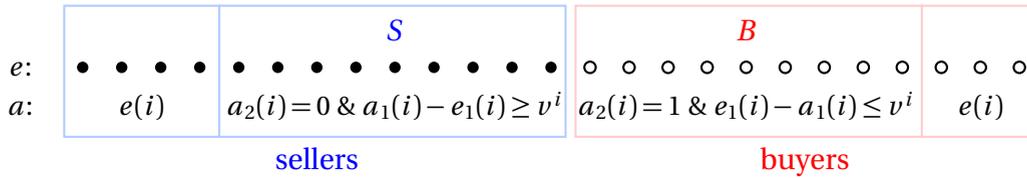


Figure 11.3 An illustration of the sets S and B in the proof of [Proposition 11.3](#).

Thus for all $i \in B$ and $j \in S$ we have $e_1(i) - a_1(i) \leq a_1(j) - e_1(j)$. By the feasibility of a , $\sum_{i \in B} (e_1(i) - a_1(i)) = \sum_{j \in S} (a_1(j) - e_1(j))$ (given that $a_1(k) = e_1(k)$ for all $k \notin B \cup S$). Thus $e_1(i) - a_1(i)$ is the same for all $i \in B$, $a_1(j) - e_1(j)$ is the same for all $j \in S$, and these amounts are equal. Denote their common value by p .

It remains to show that (p, a) is an equilibrium. Consider an individual i with $e_2(i) = 0$. If $v^i < p$ then $i \notin B$ because if $i \in B$ then $a_1(i) = e_1(i) - p$ and hence $e_1(i) > a_1(i) + v^i$ (buyer i pays more than v^i for the good) so that $e(i)$ is better for i than $a(i)$. If $v^i > p$ then $i \in B$ since otherwise $a(i) = e(i)$ and i can join with any $j \in S$ to improve upon a (i is willing to pay more than the amount j receives). Similarly for an individual i with $e_2(i) = 1$.

11.2 Exchange economy with uncertainty

People are uncertain about the future, and often believe that their wealth depends on it. To mitigate the impact of uncertainty, they engage in contracts involving payments that depend on the form the future may take. By using such contracts, they may insure each other. For example, if in future A person 1 has a high wealth and in future B he has a low wealth, and the reverse is true for person 2, then they may both be better off with a contract that transfers money from person 1 to person 2 in future A in exchange for a transfer from person 2 to person 1 in future B . We study the terms of such contracts in an equilibrium of a model like the one in [Chapter 10](#).

11.2.1 Model

We call each possible future a state of the world, or simply a state, and assume for simplicity that only two states, called 1 and 2, are possible. Every individual believes that the probability of state k is π_k , with $\pi_k > 0$. All individuals agree on these probabilities. The individuals buy and sell contracts that specify payments depending on the state that occurs. We model these contracts by stretching the notion of a good: good k is a payment of 1 unit of money if state k occurs

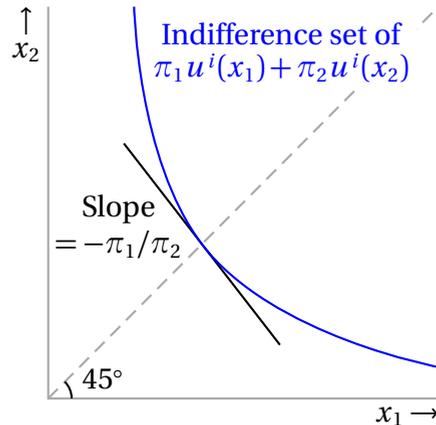


Figure 11.4 An indifference set of an individual whose preference relation is represented by a utility function $\pi_1 u^i(x_1) + \pi_2 u^i(x_2)$, where u^i is a **Bernoulli utility function**.

and nothing otherwise. Thus the owner of the bundle (x_1, x_2) obtains x_1 units of money if the state is 1 and x_2 units of money if the state is 2. Each individual i starts with the initial bundle $e(i)$.

We assume that each individual's enjoyment of the money he gets is independent of the state. Thus a bundle (x_1, x_2) is viewed by each individual as a **lottery** that gives x_1 units of money with probability π_1 and x_2 with probability π_2 . In particular, if $x_1 = x_2$ then the bundle gives the same amount in each state, and thus corresponds to a sure outcome.

The preferences over lotteries of each individual i are assumed to be represented by the expected value of a **Bernoulli utility function** u^i , so that his preference relation over the set of bundles (x_1, x_2) is represented by the utility function $U^i(x_1, x_2) = \pi_1 u^i(x_1) + \pi_2 u^i(x_2)$. We assume that each individual is **risk-averse**, so that u^i is concave, and for convenience assume also that u^i is differentiable. The **marginal rate of substitution** for individual i at (x_1, x_2) is thus $\pi_1 u_i'(x_1) / \pi_2 u_i'(x_2)$. Thus it is π_1 / π_2 if $x_1 = x_2$, greater than π_1 / π_2 if $x_1 < x_2$, and less than π_1 / π_2 if $x_1 > x_2$ as illustrated in **Figure 11.4**.

To summarize, we study the following model.

Definition 11.3: Exchange economy with uncertainty

An *exchange economy with uncertainty* $\langle N, (u^i)_{i \in N}, (\pi_1, \pi_2), e \rangle$ consists of

individuals

a finite set N

utility functions

for each individual $i \in N$, a differentiable concave function $u^i : \mathbb{R} \rightarrow \mathbb{R}$
(i 's **Bernoulli utility function**)

probabilities of states

probabilities π_1 and π_2 with $\pi_1 + \pi_2 = 1$ (π_k is the probability that each individual assigns to state k)

initial allocation

a function e that assigns to each individual i a bundle $e(i) \in \mathbb{R}^2$, the bundle that i initially owns.

The notion of equilibrium we use is an adaptation of the notion of **competitive equilibrium** for an exchange economy.

Definition 11.4: Competitive equilibrium of economy with uncertainty

A *competitive equilibrium* of the **exchange economy with uncertainty** $\langle N, (u^i)_{i \in N}, (\pi_1, \pi_2), e \rangle$ is a **competitive equilibrium** of the **exchange economy** $\langle N, (\succsim^i)_{i \in N}, e \rangle$ where \succsim^i is a preference relation represented by the utility function $\pi_1 u^i(x_1) + \pi_2 u^i(x_2)$.

11.2.2 Uncertainty about distribution of wealth

We start by considering an economy in which the total amount of money available to all individuals is independent of the state of the world, but the distribution of the money among the individuals may depend on the state. We show that if each individual is **strictly risk averse** (his Bernoulli utility function is strictly concave) then in a competitive equilibrium the individuals perfectly insure each other, consuming the same bundle in each state.

Proposition 11.4: Competitive equilibrium of economy with uncertainty

Let $\langle N, (u^i)_{i \in N}, (\pi_1, \pi_2), e \rangle$ be an **exchange economy with uncertainty** in which each function u^i is strictly concave. Assume that $\sum_{i \in N} e(i) = (c, c)$ for some $c > 0$. This economy has a unique **competitive equilibrium** (p, a) in which $p_1/p_2 = \pi_1/\pi_2$ and $a(i) = (\pi_1 e_1(i) + \pi_2 e_2(i), \pi_1 e_1(i) + \pi_2 e_2(i))$ for each individual i . That is, each individual consumes with certainty the expected amount of money he owns initially.

Proof

We first argue that (p, a) is a competitive equilibrium. We have $p_1/p_2 = \pi_1/\pi_2$, so that a bundle (x_1, x_2) is on individual i 's budget line if $\pi_1 x_1 + \pi_2 x_2 = \pi_1 e_1(i) + \pi_2 e_2(i)$. That is, all bundles on the budget line represent

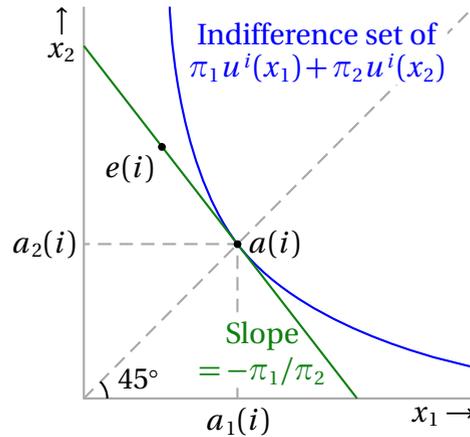


Figure 11.5 If $p_1/p_2 = \pi_1/\pi_2$ then an individual i for whom u^i is strictly concave optimally consumes the bundle $a(i)$ on his budget line for which $a_1(i) = a_2(i)$.

lotteries with the same expectation. Therefore, by i 's strict risk aversion the only optimal bundle for him is $a(i)$, which gives the amount $\pi_1 e_1(i) + \pi_2 e_2(i)$ with certainty. (See Figure 11.5.) The allocation a is feasible because for each good k

$$\sum_{i \in N} a_k(i) = \sum_{i \in N} (\pi_1 e_1(i) + \pi_2 e_2(i)) = \pi_1 \sum_{i \in N} e_1(i) + \pi_2 \sum_{i \in N} e_2(i) = \pi_1 c + \pi_2 c = c.$$

The economy has no competitive equilibrium (q, b) with $q_1/q_2 \neq \pi_1/\pi_2$. If $q_1/q_2 < \pi_1/\pi_2$ then the bundle $b(i)$ optimal for each individual i satisfies $b_1(i) > b_2(i)$, so that $\sum_{i \in N} b_1(i) > \sum_{i \in N} b_2(i)$, contradicting the feasibility condition that the total amount in each state is the same, equal to c . Similarly the economy has no equilibrium in which $q_1/q_2 > \pi_1/\pi_2$.

11.2.3 Collective uncertainty

Now suppose that state 1 is a disaster that reduces the total wealth. Then the equilibrium price ratio is greater than π_1/π_2 and every individual consumes less in state 1 than in state 2.

Proposition 11.5: Competitive equilibrium of economy with uncertainty

Let $\langle N, (u^i)_{i \in N}, (\pi_1, \pi_2), e \rangle$ be an exchange economy with uncertainty in which each function u^i is strictly concave and $\sum_{i \in N} e_1(i) < \sum_{i \in N} e_2(i)$. In a competitive equilibrium (p, a) , (i) $p_1/p_2 > \pi_1/\pi_2$ and (ii) $a_1(i) < a_2(i)$ for every individual i .

Proof

(i) If $p_1/p_2 \leq \pi_1/\pi_2$ then the bundle $x(i)$ optimal for individual i satisfies $x_1(i) \geq x_2(i)$ and thus $\sum_{i \in N} e_1(i) = \sum_{i \in N} x_1(i) \geq \sum_{i \in N} x_2(i) = \sum_{i \in N} e_2(i)$, which contradicts our assumption that the total wealth is less in state 1 than in state 2. (ii) Since the marginal rate of substitution at (x_1, x_2) with $x_1 \geq x_2$ is at most π_1/π_2 , (i) implies $a_1(i) < a_2(i)$.

11.2.4 An economy with a risk-neutral insurer

Now suppose that each individual owns one unit of wealth, which will be wiped out if state 1 occurs. The market is served by an insurer who is involved also in many other markets. The risks in each market are independent of the risks in every other market, so that the insurer faces little risk in aggregate. Thus it seems reasonable to model the insurer as acting in any given market to maximize his expected wealth $(\pi_1 x_1 + \pi_2 x_2)$. That is, we model the insurer as being risk-neutral. The next result shows that in a competitive equilibrium in such an economy the risk-averse individuals may be fully insured or only partially insured, depending on the size of the insurer's initial resources.

Proposition 11.6: Competitive equilibrium in market with insurer

Let $\langle N, (u^i)_{i \in N}, (\pi_1, \pi_2), e \rangle$ be an **exchange economy with uncertainty** in which $N = \{I\} \cup M$, where I is **risk-neutral** and all m members of M are **strictly risk-averse**, with the same strictly concave utility function u . Assume that $e(I) = (\alpha, \alpha)$ and $e(i) = (0, 1)$ for every $i \in M$.

- a. If $\alpha \geq m\pi_2$ then the economy has a unique **competitive equilibrium** (p, a) , in which $p_1/p_2 = \pi_1/\pi_2$, $a(I) = (\alpha - m\pi_2, \alpha + m\pi_1)$, and $a(i) = (\pi_2, \pi_2)$ for all $i \in M$.
- b. If $\alpha < m\pi_2$ then the economy has a competitive equilibrium. In any equilibrium $p_1/p_2 > \pi_1/\pi_2$, $a(I) = (0, \alpha(1 + p_1/p_2))$, and $a(i) = (\alpha/m, 1 - \alpha p_1/(m p_2))$ for all $i \in M$.

Proof

First note that in both cases the economy has no equilibrium with $p_1/p_2 < \pi_1/\pi_2$, since for such a price ratio we have $a_1(I) > \alpha$, so that the insurer's demand for good 1 exceeds the amount available in the economy.

a. We first show that (p, a) is the unique equilibrium with $p_1/p_2 = \pi_1/\pi_2$. For such a price system the only optimal bundle of each individual $i \in M$ is $a(i) = (\pi_2, \pi_2)$. Given $\alpha \geq m\pi_2$, in any such equilibrium feasibility requires that the insurer chooses the bundle $a(I) = (\alpha - m\pi_2, \alpha + m\pi_1)$. This bundle is on the insurer's budget line and hence is optimal for him since all bundles on his budget line yield the same expected utility. Thus the pair (p, a) is the unique equilibrium.

The economy has no equilibrium (q, b) in which $q_1/q_2 > \pi_1/\pi_2$. For such a price ratio, I 's optimal bundle is $(0, \alpha(q_1 + q_2)/q_2)$ (see the left panel in Figure 11.6) and $b_1(i) < q_2/(q_1 + q_2)$ for each $i \in M$ (see the right panel in Figure 11.6). Hence

$$b_1(I) + \sum_{i \in M} b_1(i) < m \frac{q_2}{q_1 + q_2} = m \frac{1}{1 + q_1/q_2} < m \frac{1}{1 + \pi_1/\pi_2} = m\pi_2 \leq \alpha$$

contradicting the equilibrium condition that the total demand for good 1 is equal to α .

b. The economy has no equilibrium with price ratio π_1/π_2 since then each individual $i \in M$ optimally chooses the bundle (π_2, π_2) , contradicting the feasibility requirement, given $\alpha < m\pi_2$.

Thus in any equilibrium (q, b) we have $q_1/q_2 > \pi_1/\pi_2$ and hence the insurer chooses $b(I) = (0, \alpha(q_1 + q_2)/q_2)$ (as in the left panel of Figure 11.6). The feasibility constraint then requires that $b_1(i) = \alpha/m$ for all $i \in M$. Thus the price system q is part of an equilibrium if and only if q_1/q_2 is equal to the marginal rate of substitution for each member of M at $(\alpha/m, 1 - \alpha q_1/(m q_2))$. Our assumptions ensure that at least one such price system exists. Although this result is intuitively plausible, the proof is beyond the scope of this book. The main idea of the proof is that the amount of good 1 demanded by an individual is π_2 if $q_1/q_2 = \pi_1/\pi_2$ and is close to 0 if q_1/q_2 is large enough, so that for some intermediate value of q_1/q_2 his demand is α/m .

Comment

One purpose in building and analyzing formal models is to test our intuitions about the world. The analysis may sharpen our intuition or, alternatively, suggest that our assumptions are not reasonable. Proposition 11.6 leads us to a conclusion of the latter type. In a competitive equilibrium, the insurer's profit is zero, whereas the individuals prefer the bundles they are allocated to their initial

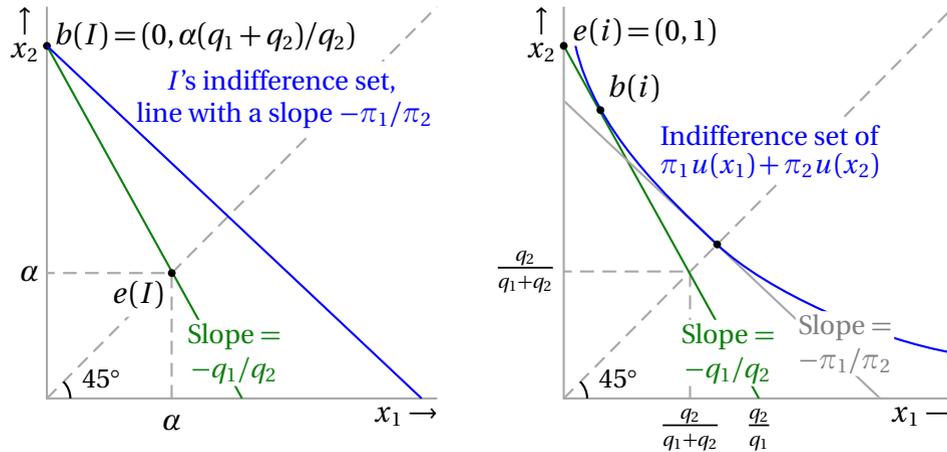


Figure 11.6 If $q_1/q_2 > \pi_1/\pi_2$ then the insurer's optimal bundle is $(0, \alpha(q_1 + q_2)/q_2)$ (left panel) and individual i 's optimal bundle $b(i)$ satisfies $b_1(i) < q_2/(q_1 + q_2)$ (right panel).

bundles. This result conflicts with our intuition that a large insurer will achieve a large profit at the expense of the risk-averse individuals. The result appears to depend on the assumption that the single large insurer takes prices as given, an assumption that does not seem reasonable. Our intuition suggests that a large monopolistic insurer will be able to exercise market power, committing to prices that generate a positive profit.

11.2.5 Heterogeneous beliefs

We have assumed so far that the probability assigned to any given state is the same for all individuals. The next example considers an economy in which the individuals' beliefs about the states differ.

Example 11.2: Exchange economy with uncertainty and heterogeneous beliefs

Consider a variant of an **exchange economy with uncertainty** in which the probabilities the individuals assign to the states differ. The set of individuals is $N = \{1, 2\}$ and each individual i 's Bernoulli utility function is $u^i(x) = x$, his initial bundle is $e(i) = (1, 1)$, and he assigns probability $\pi_1(i)$ to state 1 and $\pi_2(i)$ to state 2. We suggest you verify that the following table describes the unique competitive equilibrium for various configurations of the individuals' beliefs.

	p	$a(1)$	$a(2)$
$\frac{1}{2} < \pi_1(2) < \pi_1(1)$	$(\pi_1(2), \pi_2(2))$	$(1/\pi_1(2), 0)$	$(2 - 1/\pi_1(2), 2)$
$\pi_1(2) < \frac{1}{2} < \pi_1(1)$	$(\frac{1}{2}, \frac{1}{2})$	$(2, 0)$	$(0, 2)$
$\pi_1(2) < \pi_1(1) < \frac{1}{2}$	$(\pi_1(1), \pi_2(1))$	$(2, 2 - 1/\pi_2(1))$	$(0, 1/\pi_2(1))$

Thus when both individuals believe that state 1 is more likely than state 2 and individual 1 assigns higher probability than individual 2 to state 1 then the only equilibrium prices coincide with the probabilities that individual 2 assigns to the states, and individual 1 bets only on state 1. When the individuals disagree about the more likely state then in the unique equilibrium each of them bets on the state he believes to be more likely.

Problems

Section 11.1

1. *In equilibrium the sum of utilities is maximized.* Show that in any **competitive equilibrium** of an **exchange economy with an indivisible good and money** the sum of the individuals' utilities is maximized.
2. *Equilibrium with cash constraints.* In an **exchange economy with an indivisible good and money**, each buyer is assumed to have at least as much money as her valuation. Consider the following example of a variant of such an economy with five individuals in which some buyers have less money than their valuations. Characterize the **competitive equilibrium** of this economy under the assumption that no individual can spend more money than he originally holds.

i	1	2	3	4	5
v^i	2	10	8	4	6
$e_1(i)$	13	6	5	2	10
$e_2(i)$	0	0	0	1	1

3. *Comparative statics.* Consider an **exchange economy with an indivisible good and money** in which the unique **competitive equilibrium** price is p^* .
 - a. Show that if the valuation of one of the individuals (either a buyer or a seller) increases then any equilibrium price in the new economy is at least as high as p^* .
 - b. Show that the addition of a buyer cannot decrease the equilibrium price of the good and the addition of a seller cannot increase this price.
4. *Manipulability.* Consider an **exchange economy with an indivisible good and money** with a unique **competitive equilibrium** price. Give an example in which an individual can benefit (according to his original preferences) from acting as if he has a different valuation.

5. *Transaction costs.* Consider a variant of an **exchange economy with an indivisible good and money** in which every individual has to decide to go to the market or stay home. Going to the market involves a monetary loss of $c > 0$. A candidate for an equilibrium is now a price of the good and a profile of decisions for the individuals, where each individual has three alternatives: (i) stay home with his initial bundle; (ii) go to the market and trade at the equilibrium price; (iii) go to the market and do not trade.

Define equilibrium to be a price and a decision profile such that (a) the action of every individual is optimal, given the price and (b) the number of individuals who go to the market and buy the good is equal to the number of individuals who go to the market and sell the good.

Show that any equilibrium price is an equilibrium price in the market without transaction costs in which each seller with original valuation v has valuation $v + c$ and each buyer with original valuation v has valuation $v - c$.

6. *Payments not to participate in the market.* Construct an example of an **exchange economy with an indivisible good and money** where it is worthwhile for one of the individuals to offer other individuals the following deal: “don’t participate in the market and I will compensate you with a sum of money that will make you better off than if you refuse my offer and participate in the market”.

Section 11.2

7. *Heterogeneous beliefs.* Two individuals in an **exchange economy with uncertainty** have the same Bernoulli utility function, u , which is increasing, strictly concave, and differentiable. Individual 1 believes that the probability that the yellow basketball team will win the next game is t and individual 2 believes that this probability is s , where $0 < s \leq t < 1$. The two goods in the economy are tickets that pay \$1 if the yellow team wins and \$1 if the yellow team loses. Each individual initially has 100 tickets of each type.
- Analyze the competitive equilibrium of this market when $t = s$.
 - Assume that $t > s$. Show, graphically, that in a competitive equilibrium individual 1 holds more tickets that pay \$1 if the yellow team wins than tickets that pay \$1 if the team loses.
8. *Exchange economy with uncertainty and indivisible goods.* A show will take place only if the weather permits. To watch the show, a person needs a ticket, which will not be refunded if the show is cancelled. Each individual has a

Bernoulli utility function that takes the value $10 + m$ if the individual watches the show and m if he does not, where m is the amount of money he holds. Of the $n = n_1 + n_0$ individuals, n_1 each initially holds a single ticket and n_0 each has initially an amount of money greater than 10 but no ticket. Each individual i believes that the show will take place with probability t^i where $0 < t^n < t^{n-1} < \dots < t^2 < t^1 < 1$. Define and characterize the competitive equilibria of the variant of an **exchange economy with an indivisible good and money** that models this situation.

9. *Betting market.* Two candidates, A and B run for office. An even number n of individuals gamble on the outcome of the election. All gamblers are risk-neutral. Gambler i assigns probability α_i to A 's winning and probability $1 - \alpha_i$ to B 's winning. Each gambler chooses whether to bet on A or B . An individual who bets on A pays a price p and gets \$1 if A wins, and an individual who bets on B pays $1 - p$ and gets \$1 if B wins.
 - a. Define an equilibrium price as the price for which the number of individuals who bet on A is equal to the number of individuals who bet on B . What is a rationale for this definition?
 - b. Find the equilibrium prices if there are eight gamblers and $(\alpha_1, \dots, \alpha_8) = (0.95, 0.9, 0.8, 0.7, 0.6, 0.4, 0.1, 0)$.
10. *Time preferences.* Consider an economy with two types of individuals; each individual lives for two periods. There are n individuals of generation 1, each of whom holds \$1 in period 1, and n individuals of generation 2, each of whom holds \$1 in period 2. A bundle is a pair (x_1, x_2) with the interpretation that its holder consumes x_t units at time $t = 1, 2$. The preferences of each individual are represented by the utility function $U(x_1, x_2) = u(x_1) + \delta u(x_2)$, where $0 < \delta < 1$ and u is increasing and strictly concave.
 - a. Characterize the competitive equilibria of this economy. Are the individuals of generation 1 better off than those of generation 2, or vice versa?
 - b. (If you wish) Calculate the equilibria for the case that $u(x) = \sqrt{x}$.

Notes

The adaptation of the model of an exchange economy to an environment with uncertainty in [Section 11.2](#) was suggested by [Arrow \(1964\)](#) (originally published in French in 1953).

Problem 9 is inspired by the Iowa election markets (see <http://tippie.uiowa.edu/iem/markets/>).

