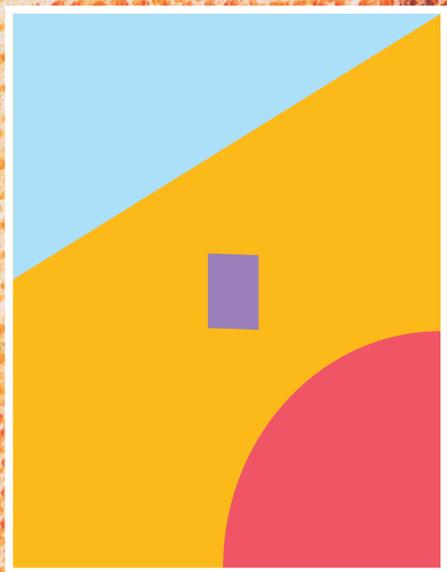


Models in Microeconomic Theory

Martin J. Osborne
Ariel Rubinstein





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1 Preferences and utility

1.1 Preferences

In the first part of the book we discuss models of individuals. These models are of interest in their own right, but we discuss them mainly to prepare for the study of interactions between individuals, which occupies the remaining parts of the book.

Our goal is to study an individual's choice from a set of alternatives in an economic environment. We can imagine building models in which the individual's characteristics are, for example, his social status or ethnic identity, his experience in the environment we are studying, or even the structure of his brain. However, we follow the approach of almost all economic theory and characterize an individual by his preferences among the alternatives, without considering the origin of these preferences.

Before we study choice, we discuss in this chapter a model of preferences over a set of alternatives. We regard an individual's preferences as a description of his mental attitude, outside the context of any choice. You may have preferences regarding the works of art shown in a local museum even though you are not going to see them; you might have preferences about what you would have done had you lived 3,000 years ago although you cannot travel in time; you might have preferences about the actions of others and the features of the natural world, like the weather, although you cannot affect these actions and features.

When we express preferences, we make statements like “I prefer a to b ”, “I like a much better than b ”, “I slightly prefer a to b ”, and “I love a and hate b ”. In this book, as in much of economic theory, the model of preferences captures only statements of the first type. That is, it contains information only about an individual's ranking of the alternatives, not about the intensity of his feelings.

At this point we suggest that you spend a few minutes completing the questionnaire at <http://gametheory.tau.ac.il/exp11/>.

We can think of an individual's preferences over a set of alternatives as encoding the answers to a questionnaire. For every pair (x, y) of alternatives in the set,

the questionnaire asks the individual which of the following three statements fits best his attitude to the alternatives.

1. I prefer x to y .
2. I prefer y to x .
3. I regard x and y as equally desirable.

The individual's mental attitude to the alternatives determines his answers to the questionnaire. We do not assume that the individual thinks explicitly about such a questionnaire; rather, his preferences reflect the answers he would give to such a questionnaire if he had to answer it.

One way to encode the individual's answers to the questionnaire is to assign a symbol "1", "-1", or "0" to (x, y) according to whether the answer is "I prefer x to y ", "I prefer y to x ", or "I regard x and y as equally desirable". However, we follow the convention in economics and describe the answers by means of a binary relation.

A binary relation on a set X specifies, for each ordered pair (x, y) of members of X , whether or not x relates to y in a certain way. For example, "acquaintance" is a binary relation on a set of people. For some pairs (x, y) of people, the statement " x is acquainted with y " is true, and for some pairs it is false. Another example of a binary relation is "smaller than" on the set of numbers. For some pairs (x, y) of numbers, x is smaller than y , and for some it is not. For a binary relation R , the expression $x R y$ means that x is related to y according to R . For any pair (x, y) of members of X , the statement $x R y$ either holds or does not hold. For example, for the binary relation $<$ on the set of numbers, we have $3 < 5$, but not $7 < 1$.

Now return to the questionnaire. One way to encode the answers to it by a binary relation is to say that x is *at least as desirable as* y , denoted $x \succcurlyeq y$, if the individual's answer to the question regarding x and y is either "I prefer x to y " or "I regard x and y as equally desirable". In this way we encode the three possible answers to the question regarding x and y , as illustrated in [Figure 1.1](#). The answer "I prefer x to y " is encoded as $x \succcurlyeq y$ but not $y \succcurlyeq x$; the answer "I prefer y to x " is encoded as $y \succcurlyeq x$ but not $x \succcurlyeq y$; and the answer "I regard x and y as equally desirable" is encoded as $x \succcurlyeq y$ and $y \succcurlyeq x$.

From the binary relation \succcurlyeq we deduce two other binary relations, \sim and \succ , defined by

$$\begin{aligned} x \sim y & \text{ if both } x \succcurlyeq y \text{ and } y \succcurlyeq x \\ x \succ y & \text{ if } x \succcurlyeq y \text{ but not } y \succcurlyeq x. \end{aligned}$$

We interpret the relation \sim as "indifference" and the relation \succ as "strict preference". These interpretations are consistent with the derivation of \succcurlyeq from the

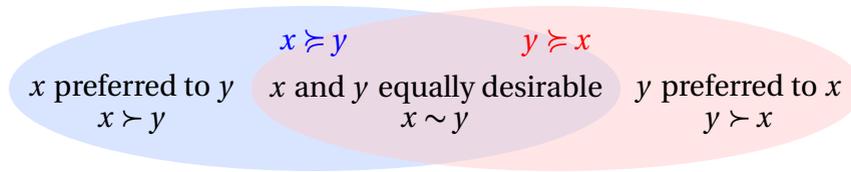


Figure 1.1 An individual's preference between x and y .

individual's answers to the questionnaire: if $x \sim y$ then $x \succcurlyeq y$ and $y \succcurlyeq x$, so that the individual's answer to the questionnaire is "I regard x and y as equally desirable", and if $x \succ y$ then the individual's answer is "I prefer x to y ".

We assume that the individual answers all the questions on the questionnaire. Given our interpretation of the binary relation \succcurlyeq as a description of responses to the questionnaire, this assumption means that for all distinct alternatives x and y either $x \succcurlyeq y$ or $y \succcurlyeq x$. We assume in addition that the same is true if x and y are the *same* alternative. That is, we assume that $x \succcurlyeq x$ for every alternative x , a property called reflexivity. The questionnaire does not ask "how do you compare x and x ?", so the reflexivity of an individual's preferences cannot be deduced from his answers. We assume it because it fits the interpretation of the binary relation: it says that the individual regards every alternative to be at least as desirable as itself.

The property that for all alternatives x and y , distinct or not, either $x \succcurlyeq y$ or $y \succcurlyeq x$, is called completeness.

Definition 1.1: Complete binary relation

A binary relation R on the set X is *complete* if for all members x and y of X , either $x R y$ or $y R x$ (or both). A complete binary relation is, in particular, *reflexive*: for every $x \in X$ we have $x R x$.

For a binary relation \succcurlyeq to correspond to a preference relation, we require not only that it be complete, but also that it be consistent in the sense that if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$. This property is called transitivity.

Definition 1.2: Transitive binary relation

A binary relation R on the set X is *transitive* if for any members x , y , and z of X for which $x R y$ and $y R z$, we have $x R z$.

In requiring that a preference relation be transitive, we are restricting the permitted answers to the questionnaire. If the individual's response to the question regarding x and y is either "I prefer x to y " or "I am indifferent between x and y ", and if his response to the question regarding y and z is "I prefer y to z " or "I am

indifferent between y and z ”, then his answer to the question regarding x and z must be either “I prefer x to z ” or “I am indifferent between x and z ”.

To conclude, we model an individual’s preferences by a complete and transitive binary relation.

Definition 1.3: Preference relation

A *preference relation* on the set X is a **complete** and **transitive** binary relation on X .

Note that the binary relations \sim (indifference) and \succ (strict preference) derived from a preference relation \succsim are both transitive. To show the transitivity of \sim , note that if $x \sim y$ and $y \sim z$ then $x \succsim y$, $y \succsim x$, $y \succsim z$, and $z \succsim y$, so by the transitivity of \succsim we have $x \succsim z$ and $z \succsim x$, and hence $x \sim z$. You are asked to show the transitivity of \succ in Problem 1a. Note also that if $x \succsim y$ and $y \succ z$ (or $x \succ y$ and $y \succ z$) then $x \succ z$ (Problem 1b).

We sometimes refer to the following additional properties of binary relations.

Definition 1.4: Symmetric and antisymmetric binary relations

A binary relation R on the set X is *symmetric* if for any members x and y of X for which $x R y$ we have $y R x$, and is *antisymmetric* if for any members x and y of X for which $x \neq y$ and $x R y$, it is not the case that $y R x$.

An example of a symmetric binary relation is “is a neighbor of” (a relation that in general is not transitive) and an example of an antisymmetric binary relation is “is older than”.

The binary relation \sim derived from a preference relation \succsim is reflexive, symmetric, and, as we have just argued, transitive. Binary relations with these properties are called equivalence relations.

Definition 1.5: Equivalence relation

A binary relation is an *equivalence relation* if it is **reflexive**, **symmetric**, and **transitive**.

Problem 4 concerns the properties of equivalence relations. In particular, it asks you to show that any equivalence relation R on a set X divides X into disjoint subsets such that two alternatives x and y belong to the same subset if and only if $x R y$. Each of these subsets is called an equivalence class. For the indifference relation, the equivalence classes are referred to also as *indifference sets*; the individual regards all alternatives in an indifference set as equally desirable and alternatives in different indifference sets as not equally desirable.

1.2 Preference formation

When we model individuals, we endow them with preference relations, which we take as given; we do not derive these preference relations from any more basic considerations. We now briefly describe a few such considerations, some of which result in preference relations and some of which do not.

Value function The individual has in mind a function v that attaches to each alternative a number, interpreted as his subjective “value” of the alternative; the higher the value, the better the individual likes the alternative. Formally, the individual’s preference relation \succsim is defined by $x \succsim y$ if and only if $v(x) \geq v(y)$. The binary relation \succsim derived in this way is indeed a preference relation: it is **complete** because we can compare any two numbers (for any two numbers a and b either $a \geq b$ or $b \geq a$ (or both)) and it is **transitive** because the binary relation \geq is transitive (if $x \succsim y$ and $y \succsim z$ then $v(x) \geq v(y)$ and $v(y) \geq v(z)$, and hence $v(x) \geq v(z)$, so that $x \succsim z$).

Distance function One alternative is “ideal” for the individual; how much he likes every other alternative is determined by the distance of that alternative from the ideal, as given by a function d . That is, the individual’s preference relation \succsim is defined by $x \succsim y$ if and only if $d(x) \leq d(y)$. This scheme is an example of a value function, with $v(x) = -d(x)$.

Lexicographic preferences An individual has in mind two **complete** and **transitive** binary relations, \succsim_1 and \succsim_2 , each of which relates to one feature of the alternatives. For example, if X is a set of computers, the features might be the size of the memory and the resolution of the screen. The individual gives priority to the first feature, breaking ties by the second feature. Formally, the individual’s preference relation \succsim is defined by $x \succsim y$ if (i) $x \succ_1 y$ or (ii) $x \sim_1 y$ and $x \succ_2 y$.

The binary relation \succsim defined in this way is a preference relation. Its completeness follows from the completeness of \succsim_1 and \succsim_2 . Now consider its transitivity. Suppose that $x \succsim y$ and $y \succsim z$. There are two cases. (i) The first feature is decisive when comparing x and y : $x \succ_1 y$. Given $y \succsim z$ we have $y \succ_1 z$, so by the transitivity of \succ_1 we obtain $x \succ_1 z$ (see Problem 1b) and thus $x \succ z$. (ii) The first feature is not decisive when comparing x and y : $x \sim_1 y$ and $x \succ_2 y$. If the first feature is decisive for y and z , namely $y \succ_1 z$, then from the transitivity of \succ_1 we obtain $x \succ_1 z$ and therefore $x \succ z$. If the first feature is not decisive for y and z , then $y \sim_1 z$ and $y \succ_2 z$. By the transitivity of \sim_1 we obtain $x \sim_1 z$ and by the transitivity of \succ_2 we obtain $x \succ_2 z$. Thus $x \succ z$.

Unanimity rule The individual has in mind n considerations, represented by the **complete** and **transitive** binary relations $\succsim_1, \succsim_2, \dots, \succsim_n$. For example, a parent

may take in account the preferences of his n children. Define the binary relation \succsim by $x \succsim y$ if $x \succsim_i y$ for $i = 1, \dots, n$. This binary relation is **transitive** but not necessarily **complete**. Specifically, if two of the relations \succsim_i disagree ($x \succsim_j y$ and $y \succ_k x$), then \succsim is not complete.

Majority rule The individual uses three criteria to evaluate the alternatives, each of which is expressed by a **complete**, **transitive**, and **antisymmetric** binary relation \succsim_i . (The antisymmetry of the relations implies that no two alternatives are indifferent according to any relation.) Define the binary relation \succsim by $x \succsim y$ if and only if a majority (at least two) of the binary relations \succsim_i rank x above y . Then \succsim is **complete**: for all alternatives x and y either $x \succsim_i y$ for at least two criteria or $y \succsim_i x$ for at least two criteria. But the relation is not necessarily **transitive**, as an example known as the Condorcet paradox shows. Let $X = \{a, b, c\}$ and suppose that $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. Then $a \succ b$ (a majority of the criteria rank a above b) and $b \succ c$ (a majority rank b above c), but $c \succ a$ (a minority rank a above c).

1.3 An experiment

The assumption that preferences are **transitive** seems natural. When people are alerted to intransitivities in their preferences they tend to be embarrassed and change their evaluations. However, it is not difficult to design an environment in which most of us exhibit some degree of intransitivity. In [Section 1.1](#) we suggested you respond to a long and exhausting questionnaire, with 36 questions, each asking you to compare a pair of alternatives taken from a set of nine alternatives. Each alternative is a description of a vacation package with four parameters: the city, hotel quality, food quality, and price.

As of April 2018, only 15% of the approximately 1,300 responses to the questionnaire do not exhibit any violation of transitivity. We count a set of three alternatives as a violation of transitivity if the answers to the three questions comparing pairs of alternatives from the set are inconsistent with transitivity. Among participants, the median number of triples that violate transitivity is 6 and the average is 9.5. (As a matter of curiosity, the highest number of intransitivities for any participant is 66. There are 84 sets of three alternatives, but the highest possible number of intransitivities is less than 84.)

A quarter of the participants' expressed preferences violate transitivity among the following alternatives.

1. A weekend in Paris, with 4 star hotel, food quality 17, for \$574.
2. A weekend in Paris, for \$574, food quality 17, with 4 star hotel.
3. A weekend in Paris, food quality 20, with 3–4 star hotel, for \$560.

Notice that options 1 and 2 describe the same package; the descriptions differ only in the order of the characteristics. Almost all participants say they are indifferent between these two alternatives, so the intransitivity is a result of differences in the expressed preferences between options 1 and 3 and options 2 and 3. That is, the order in which the features of the package are listed has an effect on the expressed preferences.

Many responses consistent with transitivity are consistent with a simple principle, like focussing on one feature, like the price, and ignoring the others, or giving priority to one feature, like the city, and breaking ties using a second feature, like the food quality (as in [lexicographic preferences](#)). Principles like these may be easier to apply consistently than more complex criteria.

1.4 Utility functions

In many economic models, an individual is described not by his preferences but by a [value function](#). This formulation does not imply that the individual explicitly derives his preferences from a value function, but only that his preferences *can* be derived from such a function. Preferences with this property are said to be *represented by* the value function. We refer to a value function that represents preferences as a *utility function*.

Definition 1.6: Utility function

For any set X and [preference relation](#) \succsim on X , the function $u : X \rightarrow \mathbb{R}$ represents \succsim if

$$x \succsim y \text{ if and only if } u(x) \geq u(y).$$

We say that u is a *utility function for* \succsim .

Example 1.1

Consider the [preference relation](#) \succsim on the set $\{a, b, c, d\}$ for which $a \succ b \sim c \succ d$. The function u for which $u(a) = 5$, $u(b) = u(c) = -1$, and $u(d) = -17$ is a utility function for \succsim .

Under what conditions can a preference relation be represented by a utility function? To answer this question, we need another definition.

Definition 1.7: Minimal and maximal alternatives

For any set X and [preference relation](#) \succsim on X , the alternative $x \in X$ is *minimal with respect to* \succsim *in* X if $y \succsim x$ for all $y \in X$ and is *maximal with respect to* \succsim *in* X if $x \succsim y$ for all $y \in X$.

The next result shows that every preference relation on a finite set has minimal and maximal members.

Lemma 1.1: Existence of minimal and maximal alternatives

Let X be a nonempty finite set and let \succsim be a preference relation on X . At least one member of X is minimal with respect to \succsim in X and at least one member is maximal.

Proof

We prove the result for minimality; the argument for maximality is analogous. We use induction on the number n of members of X . If $n = 1$ the single member of X is minimal with respect to \succsim in X . Assume the result is true for $n - 1$; we prove it is true for n . Let y be an arbitrary member of X and let x be minimal with respect to \succsim in $X \setminus \{y\}$ (a set with $n - 1$ members). If $y \succsim x$ then x is minimal in X . If not, then $x \succ y$. Take any $z \in X \setminus \{y\}$. Because x is minimal in $X \setminus \{y\}$, we have $z \succsim x$, so by transitivity $z \succsim y$. Thus y is minimal in X .

Problem 2b asks you to give an example of a preference relation on an infinite set for which there is no minimal or maximal member.

We can now show that any preference relation on a finite set can be represented by a utility function.

Proposition 1.1: Representing preference relation by utility function

Every preference relation on a finite set can be represented by a utility function.

Proof

Let X be a finite set and let \succsim be a preference relation on X . Let $Y_0 = X$ and define M_1 to be the set of alternatives minimal with respect to \succsim in Y_0 . By Lemma 1.1, Y_0 is not empty. For $k \geq 1$ inductively define $Y_k = Y_{k-1} \setminus M_k$ as long as Y_{k-1} is nonempty, and let M_{k+1} be the (nonempty) set of alternatives minimal with respect to \succsim in Y_k . In other words, at every stage remove from the set of remaining alternatives the alternatives minimal with respect to \succsim . (Figure 1.2 illustrates the construction.)

As long as Y_k is not empty, by Lemma 1.1 the set M_{k+1} is not empty. Because X is finite, for some value of K the set Y_K is empty (but the set

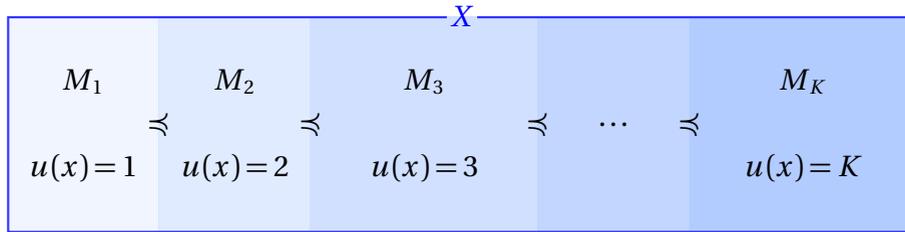


Figure 1.2 An illustration of the construction in the proof of [Proposition 1.1](#).

Y_{K-1} is nonempty). Thus every $x \in X$ is a member of some set M_k for some k , $1 \leq k \leq K$.

Define the function $u : X \rightarrow \mathbb{R}$ by $u(x) = k$ for all $x \in M_k$, $k = 1, \dots, K$. That is, attach to every alternative the number of the stage at which it is removed from X .

We argue that u is a utility function for \succsim . That is, for any alternatives a and b we have $a \succsim b$ if and only if $u(a) \geq u(b)$.

We have $u(a) = u(b)$ if and only if a and b are both minimal with respect to \succsim in $Y_{u(a)-1}$, so that $b \succsim a$ and $a \succsim b$, and hence $a \sim b$.

We have $u(b) > u(a)$ if and only if a is minimal with respect to \succsim in $Y_{u(a)-1}$, so that $b \succ a$, and $b \in Y_{u(a)-1}$ but is not minimal with respect to \succsim in $Y_{u(a)-1}$, so that it is not the case that $a \succ b$. Hence $b \succ a$.

Example 1.2: Cinema seats

A cinema has 2,000 seats, arranged in 40 rows and 50 columns. The rows are numbered starting at the screen from 1 to 40 and the columns are numbered from left to right from 1 to 50. An individual has a **lexicographic** preference relation over the set of seats. His first priority is to sit as far back as possible. Comparing seats in the same row, he prefers to sit as far to the left as possible (close to the exit, which is on the left, in case he wants to leave before the end of the screening).

In the construction in the proof of [Proposition 1.1](#), the set M_1 consists of the single seat in row 1, column 50, so this seat is assigned the utility 1; the set M_2 consists of the single seat in row 1, column 49, so this seat is assigned the utility 2; ...; the set M_{2000} consists of the single seat in row 40, column 1, so this seat is assigned the utility 2,000. (A cinema with ten rows of ten seats is illustrated in [Figure 1.3](#).) The individual's preference relation is represented by the utility function u defined by $u(x) = 50r(x) - c(x) + 1$, where $r(x)$ is the row number of the seat and $c(x)$ is its column number.

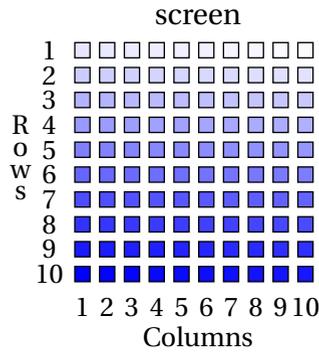


Figure 1.3 A cinema like the one in [Example 1.2](#), with ten rows of ten seats. For the individual, no two seats are indifferent. He prefers seat x to seat y if x is shaded with a darker blue than y .

Many preference relations on *infinite* sets can also be represented by utility functions. A simple example is the preference relation \succsim on the set of nonnegative real numbers defined by $x \succsim y$ if and only if $x \geq y$, which is represented by the utility function u defined by $u(x) = x$. However, not all preference relations on infinite sets can be represented by utility functions. An example is the [lexicographic preference relation](#) over the unit square $X = \{(x_1, x_2) : x_1, x_2 \in [0, 1]\}$ for which the first priority is the first coordinate and the second priority is the second coordinate (so that, for example, $(0.3, 0.1) \succ (0.2, 0.9) \succ (0.2, 0.8)$). ([Figure 1.4](#) shows the set of alternatives preferred to a given alternative.)

Proposition 1.2: Preference relation not represented by utility function

The (lexicographic) [preference relation](#) \succsim on $\{(x_1, x_2) : x_1, x_2 \in [0, 1]\}$ defined by $(x_1, x_2) \succ (y_1, y_2)$ if and only if either (i) $x_1 > y_1$ or (ii) $x_1 = y_1$ and $x_2 > y_2$ is not [represented by any utility function](#).

The proof of this result requires more mathematical knowledge than the other arguments in the book.

Proof

Assume, contrary to the claim, that the function u represents \succsim . For each $x \in [0, 1]$, we have $(x, 1) \succ (x, 0)$, so that $u(x, 1) > u(x, 0)$. Define a function f that assigns to every number $x \in [0, 1]$ a rational number in the interval $(u(x, 0), u(x, 1))$. Such a number exists because between any two real numbers there is a rational number. The function f is one-to-one since if $a > b$ then $(a, 0) \succ (b, 1)$, so that $u(a, 0) > u(b, 1)$, and hence the interval $(u(a, 0), u(a, 1))$ from which $f(a)$ is selected does not intersect the interval $(u(b, 0), u(b, 1))$ from which $f(b)$ is selected. The contradiction

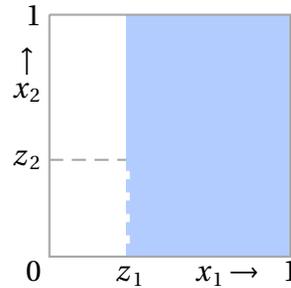


Figure 1.4 The set of alternatives preferred to (z_1, z_2) according to the lexicographic preference relation described in the text is the area shaded blue, excluding the part of the boundary indicated by a dashed line.

now follows from Cantor's diagonal argument, which shows that there is no one-to-one function from the set $[0, 1]$ into a countable set (like the set of rational numbers).

If a utility function represents a given preference relation, then many other utility functions do so too. For example, if the function u represents a given preference relation then so does the function $3u - 7$ or any other function of the form $au + b$ where a is a positive number. Generally, we have the following result. Note that we define a function f to be increasing if $f(x) > f(y)$ whenever $x > y$ (and nondecreasing if $f(x) \geq f(y)$ whenever $x > y$).

Proposition 1.3: Increasing function of utility function is utility function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. If u represents the preference relation \succsim on X , then so does the function w defined by $w(x) = f(u(x))$ for all $x \in X$.

Proof

We have $w(x) \geq w(y)$ if and only if $f(u(x)) \geq f(u(y))$ if and only if $u(x) \geq u(y)$ (given that f is increasing), which is true if and only if $x \succsim y$.

Problems

1. *Properties of binary relations.* Assume that \succsim is a preference relation.
 - a. Show that the binary relation \succ defined by $x \succ y$ if $x \succsim y$ and not $y \succsim x$ is transitive and antisymmetric.
 - b. Show that if $x \succ y$ and $y \succ z$ (or $x \succ y$ and $y \succsim z$) then $x \succ z$.

2. *Minimal element.* Let \succsim be a **preference relation** over a finite set X .
 - a. Show that a is **minimal** with respect to \succsim in X if and only if there is no $x \in X$ such that $a \succ x$.
 - b. Give an example to show that if X is not finite then a preference relation may have no **minimal and maximal** elements in X .
3. *Similarity relations.* Consider the following **preference formation scheme**. An individual has in mind a function $v : X \rightarrow \mathbb{R}$ that attaches to each alternative a number, but is sensitive only to significant differences in the value of the function; he is indifferent between alternatives that are “similar”. Specifically, the individual prefers x to y if $v(x) - v(y) > 1$ and is indifferent between x and y if $-1 \leq v(x) - v(y) \leq 1$. Is the individual’s **preference relation** necessarily **transitive**?
4. *Equivalence relations.*
 - a. Give two examples of **equivalence relations** on different sets.
 - b. Show that the binary relation R on the set of positive integers defined by $x R y$ if $x + y$ is even is an **equivalence relation**.
 - c. A partition of the set X is a set of nonempty subsets of X such that every member of X is a member of one and only one subset. For example, the set of sets $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ is a partition of the set $\{1, 2, 3, 4, 5, 6\}$. Show that every **equivalence relation** R on X induces a partition of the set X in which x and y are in the same member of the partition if and only if $x R y$.
5. *Independence of properties.* Find an example of a binary relation that is **complete** and **transitive** but not **symmetric**. Find also an example of a binary relation that is reflexive, transitive, and symmetric but not complete.
6. *Shepard scale and Escher.* Listen to the Shepard scale and look at a picture of Penrose stairs. (The video at <http://techchannel.att.com/play-video.cfm/2011/10/10/AT&T-Archives-A-Pair-of-Paradoxes> combines them. The lithograph *Ascending and descending* by M. C. Escher is a rendering of Penrose stairs.) Explain the connection between these two examples and the concept of transitivity.
7. *Utility representation.* Let X be the set of all positive integers.
 - a. An individual prefers the number 8 to all other numbers. Comparing a pair of numbers different from 8 he prefers the higher number. Construct a utility function that represents these preferences.

- b.* An individual prefers the number 8 to all other numbers. Comparing a pair of numbers different from 8 he prefers the number that is closer to 8. Construct a utility function that represents these preferences.
8. *Utility representation.* Consider the [preference relation](#) on the positive integers in which x is preferred to y if *either* (i) x is even and y is odd, *or* (ii) x and y are even and $x > y$, *or* (iii) x and y are odd and $x > y$.
- a.* Show that no utility function with integer values represents this preference relation.
- b.* Define a utility function the values of which are real numbers that represents the preference relation.
9. *Representations with additive utility.* An individual has preferences over the set of units in an apartment building. He prefers a unit with 5 rooms on floor 12 to one with 4 rooms on floor 20, one with 4 rooms on floor 5 to one with 2 rooms on floor 12, and one with 2 rooms on floor 20 to one with 5 rooms on floor 5.
- a.* Show that the individual's preference are consistent with a [preference relation](#).
- b.* Show that the individual's preference relation cannot be represented by a function u with $u(x) = f(r(x)) + g(l(x))$ for functions f and g , where $r(x)$ is the number of rooms and $l(x)$ is the floor for any unit x .

Notes

The formalization of the notion of a preference relation appears to be due to Frisch, in 1926 (see [Frisch 1957](#)), and the first analysis of the problem of representing a preference relation by a utility function appears to be [Wold \(1943\)](#). [Proposition 1.2](#) is due to [Debreu \(1954, footnote 1, 164\)](#). The exposition of the chapter draws on [Rubinstein \(2006a, Lecture 1\)](#).

